Some Questions in the Theory of Moments

by

N. I. Ahiezer
M. Krein

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О НЕКОТОРЫХ ВОПРОСАХ ТЕОРИИ МОМЕНТОВ

Н. АХИЕЗЕР и М. КРЕЙН

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FOREWORD

The present book deals with several specialized questions pertaining to the so-called moment problem which have for one reason or another been of interest to the authors.

The book is divided into separate articles which are basically independent of one another; however, it should not be thought that these articles are not connected with one another. On the contrary, there is a close connection and the articles are arranged in a discernible logical order.

The basic article (the L-moment problem) has as its starting point several (little-known) ideas and problems advanced by the late academician A. A. Markov [13b, c, d]. [1] In a series of papers [1a–i] the authors have completed and generalized A. A. Markov's results, connecting them with present-day theories apparently unknown to A. A. Markov.

The three articles which follow treat in the light of functional analysis some of the questions solved or discussed in the first article.

The subsequent two articles are devoted to various applications of one class of functions (named N-functions by the authors) which play a large role in the moment problem.

As is usually the case, the authors propose to write at least one more book which will deal with all the important questions of the theory which have not found a place in this book.

They hope, therefore, that no one will reproach them because questions of determinateness of the moment problem remain unanswered.

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1) The numbers in square brackets refer to the bibliography located at the end of the book.

THE L-PROBLEM OF MOMENTS

N. AHIEZER AND M. KREĬN

CHAPTER I

THE FUNCTIONALS $\mathfrak{S}$ AND $\mathfrak{C}$

Let there be given a sequence of real numbers

$$s_0, s_1, s_2, \cdots, s_n; \tag{s}$$

in the space of (generally speaking complex) polynomials

$$G_n(u) = A_0 + A_1 u + \cdots + A_n u^n$$

of degree not greater than n we define the functional $\mathfrak{S}$ by setting

$$\mathfrak{S}\{G_n\} = A_0 s_0 + A_1 s_1 + \cdots + A_n s_n.$$

Let us agree to call a sequence (s) *positive definite* (or *non-negative definite*) in a certain interval J of the axis $-\infty < u < \infty$ if from the relations

$$G_n(u) \not\equiv 0, \quad G_n(u) \geq 0 \qquad (u \subset J)$$

always follows the inequality $\mathfrak{S}\{G_n\} > 0$ (or $\mathfrak{S}\{G_n\} \geq 0$).

In a similar way, if a sequence of complex numbers is given

$$c_0 \gtrless 0, \quad c_1, c_2, \cdots, c_n, \tag{c}$$

we define in the space of quasi-polynomials

$$T_n(z) = \sum_{k=-n}^{n} A_k z^k, \qquad (z = e^{it})$$

of "degree" not greater than n the functional $\mathfrak{C}$ by setting

$$\mathfrak{C}\{T_n\} = \sum_{k=-n}^{n} A_k c_k,$$

where

$$c_{-k} = \bar{c}_k \qquad (k = 1, 2, \cdots, n)$$

(the bar denotes change to the complex conjugate value).

It is obvious that if $T_n(e^{it})$ is a real trigonometric polynomial ($A_{-k} = \overline{A}_k$, $k = 0, 1, \cdots, n$), then every value of the functional $\mathfrak{S}\{T_n\}$ is a real number.

Let us agree to call the sequence (c) *positive definite* (or *non-negative definite*) on a certain arc B of the circumference $0 \leq t \leq 2\pi$, if from the relations

$$T_n(z) \not\equiv 0, \quad T_n(e^{it}) \geq 0 \qquad (t \subset B)$$

always follows the inequality

$$\mathfrak{S}\{T_n\} > 0 \quad (\text{or } \mathfrak{S}\{T_n\} \geq 0).$$

It is easy to construct examples of positive definite and non-negative definite sequences.

1. Let us take any numbers

$$\rho_1 > 0, \ \rho_2 > 0, \cdots, \rho_m > 0$$

$$\xi_1 < \xi_2 < \cdots < \xi_m$$

and put

$$s_k = \rho_1 \xi_1^k + \rho_2 \xi_2^k + \cdots + \rho_m \xi_m^k \qquad (k = 0, 1, 2, \cdots). \tag{*}$$

If

$$G_n(u) = A_0 + A_1 u + \cdots + A_n u^n,$$

then

$$\mathfrak{S}\{G_n\} = \rho_1 G_n(\xi_1) + \rho_2 G_n(\xi_2) + \cdots + \rho_m G_n(\xi_m).$$

Therefore for $a < \xi_1$, $b > \xi_m$, the sequence

$$s_0, s_1, \cdots, s_n$$

is positive definite in the interval $<a, b>$ if $n \leq 2m - 1$ and non-negative definite if $n > 2m - 1$. While the same sequence in the interval $<\xi_1, \xi_m>$ will be positive definite only for $n \leq 2m - 3$, it remains non-negative definite for $n > 2m - 3$.

2. Let us take in an interval (a, b) a real integrable function $p(x) \geq 0$, for which $\int_a^b p(x)dx > 0$, and put

$$s_k = \int_a^b x^k p(x)dx \qquad (k = 0, 1, 2, \cdots, n)$$

(assuming in case of an infinite interval (a, b) that all integrals introduced make sense).

Since

$$\mathfrak{S}\{G_n\} = \int_a^b G_n(x)\, p(x)\, dx,$$

the sequence

$$s_0, s_1, \cdots, s_n$$

is positive definite in the interval (a, b).

In a similar way one can construct positive and negative sequences (c) on an arc of the circumference.

By using Stieltjes integrals both of the above examples can be combined into one, if we set

$$s_k = \int_a^b x^k d\sigma(x) \qquad (k = 0, 1, 2, \cdots), \tag{**}$$

where $\sigma(x)$ is a bounded non-decreasing function.

In what follows, we shall see that every finite sequence (s) non-negative definite in the interval (a, b) $(-\infty \leq a < b \leq \infty)$ admits a representation (**). M. Riesz [37] gave a particularly simple and elegant proof of this fact.

In the present chapter we concern ourselves with the study of positive and non-negative definite sequences (s) and (c), i.e., of functionals $\mathfrak{S}$ and $\mathfrak{C}$, by a purely algebraic method which is a further development of methods of Stridsberg [40] and M. Riesz [37], paying particular attention to the representation of sequences by sums[1] with the minimal number of summands (canonical representations).

§ 1

1. We begin by establishing a criterion for positivity (or non-negativity) of a sequence

$$s_0, s_1, s_2, \cdots, s_n \tag{1}$$

in the interval $(-\infty, \infty)$.

Theorem 1. *In order that the sequence* (1) *be positive definite (or non-negative definite) in the interval* $(-\infty, \infty)$ *it is necessary and sufficient that the quadratic form of Hankel*

1) Of the form (*), if the sequence being discussed is (s).

$$\sum_{0}^{m} s_{i+k} X_i X_k,$$

where $m = \left[\frac{n}{2}\right]$, *be positive* (*or non-negative*).

The proof follows from the fact that a polynomial $G_n(u)$ of degree not more than n will be non-negative in the interval $(-\infty, \infty)$ if and only if it admits a representation

$$G_n(u) = \left| \sum_{k=0}^{m} (x_k + iy_k) u^k \right|^2 = \sum_{0}^{m} x_\lambda x_\mu u^{\lambda+\mu} + \sum_{0}^{m} y_\lambda y_\mu u^{\lambda+\mu} \quad \left(m = \left[\frac{n}{2}\right]\right),$$

where x_k, y_k $(k = 0, 1, \cdots, m)$ are real numbers.

2. Let

$$s_0, s_1, s_2, \cdots, s_{2m-2} \tag{2}$$

be a positive definite sequence in the interval $(-\infty, \infty)$.

By Theorem 1 the following inequality holds:

$$D_k = \begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_k & s_{k+1} & \cdots & s_{2k} \end{vmatrix} > 0 \qquad (k = 0, 1, 2, \ldots, m-1).$$

Putting

$$P_k(u) = \begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_{k-1} & s_k & \cdots & s_{2k-1} \\ 1 & u & \cdots & u^k \end{vmatrix} \qquad (k = 1, 2, \ldots, m-1), \quad P_0(u) \equiv 1,$$

we find $P_k(u)$ is a polynomial of exactly kth degree, satisfying the conditions

$$\mathfrak{S}\{P_k(u) u^i\} = 0 \qquad (i = 0, 1, \cdots, k-1;\ k \leq m-1)$$

and called an *orthogonal polynomial* of degree k.

Taking

$$s_k = \int_a^b x^k p(x)\, dx \qquad (k = 0, 1, 2, \cdots)$$

we obtain the classical systems of orthogonal polynomials:

1. Jacobi's for $a = -1$, $b = 1$, $p(x) = (1+x)^{\lambda}(1-x)^{\mu}$ $(\lambda > -1,\ \mu > -1)$.

2. Laguerre's for $a = 0$, $b = \infty$, $p(x) = e^{-x}$.

3. Hermite's for $a = -\infty$, $b = \infty$, $p(x) = e^{-x^2}$.

It is easily seen that every orthogonal polynomial of degree k equals $\alpha P_k(u)$, where α is a constant. In fact, the coefficients A_i of the orthogonal polynomial

$$A_0 + A_1 u + \cdots + A_k u^k$$

satisfy the equations

$$A_0 s_0 + A_1 s_1 + \cdots + A_k s_k = 0,$$

$$A_0 s_1 + A_1 s_2 + \cdots + A_k s_{k+1} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_0 s_{k-1} + A_1 s_k + \cdots + A_k s_{2k-1} = 0,$$

and the rank of this system equals k, since $D_{k-1} > 0$.

The polynomial

$$\widehat{P}_k(u) = \frac{1}{\sqrt{D_{k-1} D_k}} P_k(u) = \sqrt{\frac{D_{k-1}}{D_k}}\, u^k + \cdots,$$

satisfying not only the conditions of orthogonality but also the condition of normality

$$\mathfrak{S}\{\hat{P}_k^2\} = 1$$

is called the *ortho-normalized polynomial* of kth degree.

Following M. Riesz [37], we shall call a polynomial $Q_k(u) \not\equiv 0$, which is of degree k, a *quasi-orthogonal* polynomial of rank k, if

$$\mathfrak{S}\{Q_k(u)\, u^i\} = 0 \qquad (i = 0, 1, 2, \cdots, k-2;\ k \leq m).$$

It is apparent that the polynomials $P_k(u)$, $P_{k-1}(u)$ are quasi-orthogonal polynomials of rank k. For any quasi-orthogonal polynomial $Q_k(u)$ of rank $k \leq m-1$ it is possible to select an α such that the difference $Q_k(u) - \alpha P_k(u)$ has degree $\leq (k-1)$ and thus is an orthogonal polynomial of degree $k-1$ or identically zero. It follows from this that every quasi-orthogonal polynomial $Q_k(u)$ of rank $k \leq m-1$ may be represented in the form

$$Q_k(u) = \alpha P_k(u) + \beta P_{k-1}(u),$$

where the constants (complex) α, β are not both zero.

It is easily seen that every quasi-orthogonal polynomial $Q_k(u)$ of rank $k \leq m$ may also be written in the form of a determinant

$$Q_k(u) = \begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_{k-2} & s_{k-1} & \cdots & s_{2k-2} \\ \alpha_0 & \alpha_1 & \cdots & \alpha_k \\ 1 & u & \cdots & u^k \end{vmatrix}$$

where $\alpha_0, \alpha_1, \cdots, \alpha_k$ are constants.

Theorem 2. *All roots of any real quasi-orthogonal polynomial are real and simple.*

Proof. Let $\xi_1, \xi_2, \cdots, \xi_p$ be all the distinct real roots of odd multiplicity of the real quasi-orthogonal polynomial $Q_k(u)$ of rank k, so that the product

$$G(u) = \epsilon(u - \xi_1)(u - \xi_2) \cdots (u - \xi_p) Q_k(u) \not\equiv 0 \qquad (\epsilon = \pm 1)$$

is non-negative for all real u.

If we assume that $p \leq k - 2$, we find by virtue of the characteristic property of the quasi-orthogonal polynomial that

$$\mathfrak{S}\{G\} = 0,$$

which is impossible, since

$$G(u) \geq 0.$$

Therefore $p \geq k - 1$, which means that $\xi_1, \xi_2, \cdots, \xi_p$ are simple roots of the polynomial $Q_k(u)$; but since it is real, p equals the degree of the polynomial $Q_k(u)$.

3. Let $\xi_1, \xi_2, \cdots, \xi_m$ be all the roots of the real quasi-orthogonal polynomial $Q_m(u)$ of mth degree.

An arbitrary polynomial $G(u)$ of degree $\leq 2m - 2$ may be represented in the form

$$G(u) = Q_m(u)\, q(u) + r(u),$$

where $q(u)$ is a polynomial of degree $\leq m - 2$, and

$$r(u) = \sum_{k=1}^{m} \frac{Q_m(u)}{(u-\xi_k)\,Q_m'(\xi_k)}\, r(\xi_k) = \sum_{k=1}^{m} \frac{Q_m(u)}{(u-\xi_k)\,Q_m'(\xi_k)}\, G(\xi_k)$$

is a polynomial of degree $\leq m-1$.

By virtue of the characteristic property of the quasi-orthogonal polynomial $Q_m(u)$:

$$\mathfrak{S}\{G\} = \mathfrak{S}\{r\},$$

so that the so-called *quadrature formula*[1)]

$$\mathfrak{S}\{G\} = \sum_{k=1}^{m} \rho_k G(\xi_k) \tag{3}$$

holds, where the numbers ρ_k $(k = 1, 2, \cdots, m)$ are defined by the formula

$$\rho_k = \mathfrak{S}\left\{\frac{Q_m(u)}{(u-\xi_k)\,Q_m'(\xi_k)}\right\}. \tag{3$_1$}$$

Substituting in formula (3)

$$G(u) = q_k^2(u) = \left[\frac{Q_m(u)}{(u-\xi_k)\,Q_m'(\xi_k)}\right]^2,$$

we find that

---

1) The basis for this name is the fact that for

$$s_k = \int_a^b x^k p(x)\,dx \qquad (k = 0, 1, 2, \cdots)$$

formula (3) takes the form

$$\int_a^b G(x)\,p(x)\,dx = \sum_{k=1}^{m} \rho_k G(\xi_k)$$

and being exact when $G(x)$ is a polynomial of degree $\leq 2m-2$, it may serve for approximate calculation of the integral

$$\int_a^b G(x)\,p(x)\,dx$$

when $G(x)$ is not a polynomial of degree $2m-2$.

$$\rho_k = \mathfrak{S}\{q_k^2(u)\} > 0 \qquad (k = 1, 2, \cdots, m),$$

since the sequence (2) is positive definite.

In particular, setting $G(u) = u^k$ $(k = 0, 1, \cdots, 2m-2)$, we obtain the first part of the following proposition:

Theorem 3. a) *If a sequence* $s_0, s_1, \cdots, s_{2m-2}$ *is positive definite in the interval* $(-\infty, \infty)$, *there exist infinitely many "canonical" representations*

$$s_k = \sum_{i=1}^{m} \rho_i \xi_i^k \qquad (k = 0, 1, 2, \cdots, 2m-2), \tag{4}$$

where

$$\rho_j > 0 \ (j = 1, 2, \cdots, m), \ -\infty < \xi_1 < \xi_2 < \cdots < \xi_m < \infty.$$

b) *The systems of numbers* $\xi_1, \xi_2, \cdots, \xi_m$ *in two distinct representations* (4) *alternate.*

c) *Among the representations* (4) *there exists one and only one representation in which the system of numbers* $\xi_1, \xi_2, \cdots, \xi_m$ *contains an arbitrary given point* η, *which is not a root of the polynomial* $P_{m-1}(u)$.

d) *Given any real number* s_{2m-1}, *among the representation* (4) *we can find one and only one such that*

$$s_{2m-1} = \sum_{i=1}^{m} \rho_i \xi_i^{2m-1}. \tag{4'}$$

Proof. The assertion a) is already proved.

Let us prove assertion b).

Let

$$s_k = \sum_{i=1}^{m} \rho_i' \xi_i'^{\,k} \qquad (k = 0, 1, \cdots, 2m-2) \tag{5}$$

be one of the representations (4). Let us construct the polynomial

$$G(u) = \frac{1}{(u-\xi_k')(u-\xi_{k+1}')} \prod_{1}^{m} (u-\xi_i')^2.$$

It then follows from (5) and (3) that

$$\mathfrak{S}\{G\} = \sum_{i=1}^{m} \rho_i' G(\xi_i') = 0,$$

and consequently for any representation (4):

$$\mathfrak{S}\{G\} = \sum_{i=1}^{m} \rho_i G(\xi_i) = 0. \qquad (5_1)$$

Since

$$G(u) > 0 \quad \text{for} \quad u < \xi_k' \quad \text{and} \quad u > \xi_{k+1}',$$

$$G(u) < 0 \quad \text{for} \quad \xi_k' < u < \xi_{k+1}',$$

it follows from (5_1) that between two consecutive points ξ_k', ξ_{k+1}' of representation (5) there always lies at least one point ξ_j of every other representation, from which assertion b) follows.

If $Q_m(u)$ is a real quasi-orthogonal polynomial of degree m, then for $P_{m-1}(\eta) \neq 0$ the polynomial

$$Q_m(u;\ \eta) = Q_m(u)P'_{m-1}(\eta) - P_{m-1}(u)Q_m(\eta)$$

also has these properties and has η as a root. Thus, to the polynomial $Q_m(u;\eta)$ there corresponds a representation (4) in which one of the numbers ξ_j $(j = 1, 2, \cdots, m)$ equals η. The uniqueness of this representation follows from part b). Thus assertion c) is verified.

Only assertion d) remains to be proved. If we add an arbitrary real number s_{2m-1} to the sequence (2) the functional $\mathfrak{S}$ becomes meaningful for the set of all polynomials of degree $\leq 2m - 1$. If we then take the orthogonal polynomial $P_m(u)$ as the polynomial $Q_m(u)$ in the quadrature formula we find that the corresponding quadrature formula (3) holds for polynomials of degree $\leq 2m - 1$. Substituting into this result $G(u) = u^k$ $(k = 0, 1, \cdots, 2m - 1)$, we obtain the required representation (4), (4′).

Conversely, let representation (4), (4′) hold. Then the quadrature formula (3) will be valid for polynomials $G(u)$ of degree $\leq 2m - 1$. If we here set

$$G(u) = u^i R(u), \quad \text{where } R(u) = \prod_{k=1}^{m} (u - \xi_k) \qquad (i = 0, 1, \cdots, m - 1),$$

we find that $R(u)$ is an orthogonal polynomial of degree m, i.e., $R(u) = \alpha P_m(u)$. Thus, the numbers $\xi_1, \xi_2, \cdots, \xi_m$ are uniquely determined from (4) and (4′); and as soon as they are determined, the numbers ρ_j $(j = 1, 2, \cdots, m)$ are fixed by formula (3_1) after substitution in it of $P_m(u)$ for $Q_m(u)$.

Of interest is

Theorem 4. *The roots of any two arbitrary real, linearly independent, quasi-orthogonal polynomials of the same rank alternate.*

Proof. By assertion b) of Theorem 3 the roots of any two linearly independent real quasi-orthogonal polynomials $Q_m(u)$ and $Q_m^*(u)$ of degree and rank m alternate. It remains to show that the roots of $P_{m-1}(u)$ alternate with the roots of an arbitrary real quasi-orthogonal polynomial $Q_m(u)$ of degree and rank m. Let us represent $P_{m-1}(u)$ in the form $P_{m-1}(u) = \alpha Q_m(u) + \beta Q_m^*(u)$ $(\beta \neq 0)$; then for the roots $\xi_1 < \xi_2 < \cdots < \xi_m$ of the polynomial $Q_m(u)$, we have $P_{m-1}(\xi_i) = \beta Q_m^*(\xi_i)$. Since the numbers $Q_m^*(\xi_i)$ and $Q_m^*(\xi_{i+1})$ $(i = 1, 2, \cdots, m-1)$ are of opposite sign, the numbers $P_{m-1}(\xi_i)$ and $P_{m-1}(\xi_{i+1})$ $(i = 1, 2, \cdots, m-1)$ are also of opposite sign. Thus, between ξ_i and ξ_{i+1} $(i = 1, 2, \cdots, m-1)$ there lies one and only one root of the polynomial $P_{m-1}(u)$, as was required to prove.

4. We shall now drop the assumption that the sequence

$$s_0 > 0,\ s_1,\ \cdots,\ s_{2m-2} \tag{2}$$

is positive definite in the interval $(-\infty, \infty)$ and suppose that this sequence is non-negative definite.

Let D_{p-1} be the last non-zero element in the sequence

$$D_0,\ D_1,\ D_2,\ \cdots,\ D_{m-2},$$

i.e., let [1)]

$$D_0 > 0,\ D_1 > 0,\ \cdots,\ D_{p-1} > 0,\ D_p = D_{p+1} = \cdots = D_{m-2} = 0.$$

By Theorem 3 the representation

$$s_k = \sum_{i=1}^{p} \rho_i \xi_i^k \qquad (k = 0, 1, \cdots, 2p-1)$$

holds, where $\xi_1 < \xi_2 < \cdots < \xi_p$ are roots of the orthogonal polynomial $P_p(u)$. We shall prove that for $p \leq m-2$ the equation

$$N_j = \mathfrak{S}\{P_p^2(u)\, u^j\} = 0 \qquad (j = 0, 1, \cdots, 2m-2p-3) \tag{6_1}$$

holds.

---

1) The possibility $p = m-1$ is not excluded. In this case, $D_p = D_{p+1} = \cdots = D_{m-2} = 0$ drops out automatically.

Since for any real λ:

$$A_k(u) = u^{2k}(1+\lambda u)^2 P_p^2(u) \geqslant 0 \quad (k=0,1,\ldots,m-p-2),$$

$$B_i(u) = u^{2i}(1+\lambda u^2)^2 P_p^2(u) \geqslant 0 \ (i=0,1,\ldots,m-p-3),$$

we have

$$0 \leqslant \mathfrak{S}\{A_k(u)\} = N_{2k} + 2\lambda N_{2k+1} + \lambda^2 N_{2k+2};$$

$$0 \leqslant \mathfrak{S}\{B_i(u)\} = N_{2i} + 2\lambda N_{2i+2} + \lambda^2 N_{2i+4}.$$

Setting $k = i = 0$ in these relations and noting that $D_p = 0$ and

$$N_0 = \mathfrak{S}\{P_p^2(u)\} = D_{p-1}\mathfrak{S}\{P_p(u)u^p\} = D_{p-1}D_p, \tag{6_2}$$

we find that $N_1 = N_2 = 0$. Therefore, setting $k = i = 1$, we obtain $N_3 = N_4 = 0$, and continuing this argument we find

$$N_0 = N_1 = \cdots = N_{2m-2p-4} = 0.$$

Finally, taking $k = m - p - 2$, we obtain from the first of the above inequalities

$$N_{2m-2p-3} = 0, \quad N_{2m-2p-2} \geq 0.$$

Thus our assertion is verified.

Theorem 5 (E. Fischer [25]). *If a sequence $s_0 > 0$, $s_1, \cdots, s_{2m-2}$ is non-negative in the interval $(-\infty, \infty)$ and if D_{p-1} is the last non-zero element in the sequence*

$$D_0, D_1, \cdots, D_{m-2},$$

then there exists a unique representation

$$s_k = \sum_{i=1}^{p} \rho_i \xi_i^k \qquad (k = 0, 1, 2, \ldots, 2m-3)$$

$$s_{2m-2} = \sum_{i=1}^{p} \rho_i \xi_i^{2m-2} + M, \tag{7}$$

where

$$\xi_1 < \xi_2 < \cdots < \xi_p; \quad \rho_i > 0 \quad (i = 1, 2, \cdots, p) \quad \text{and} \quad M \geq 0.$$

If $p = m - 1$, then $M = \dfrac{D_{m-1}}{D_{m-2}}$.

Proof. Let

$$G(u) = Au^{2m-2} + Bu^{2m-3} + \cdots$$

be an arbitrary polynomial of degree $\leq 2m-2$. We reduce it to the form $G(u) = \frac{P_p^2(u)}{D_{p-1}^2}\{Au^{2m-2p-2} + Bu^{2m-2p-3} + \cdots\} + H(u)$, where $H(u)$ is a polynomial of degree $\leq 2p-1$. By (6_1) we have

$$\mathfrak{S}\{G(u)\} = AM + \mathfrak{S}\{H(u)\},$$

where

$$M = \frac{1}{D_{p-1}^2}\mathfrak{S}\{P_p^2(u)\, u^{2m-2p-2}\} = \frac{N_{2m-2p-2}}{D_{p-1}^2} > 0.$$

Noting that $H(\xi_i) = G(\xi_i)$, where $\xi_1, \xi_2, \cdots, \xi_p$ are roots of the orthogonal polynomial $P_p(u)$ and expanding $\mathfrak{S}\{H(u)\}$ by the quadrature formula, we find that

$$\mathfrak{S}\{G\} = MA + \sum_{i=1}^{p} \rho_i G(\xi_i).$$

Letting $G(u) = u^k$ $(k = 0, 1, \cdots, 2m-2)$ we obtain representation (7).

If $p = m-1$, $M = \frac{N_0}{D_{p-1}^2} = \frac{D_p}{D_{p-1}} = \frac{D_{m-1}}{D_{m-2}}$, by (6_2). The uniqueness of representation (7) follows from the fact that by Theorem 3 the representation

$$s_k = \sum_{i=1}^{p} \rho_i \xi_i^k \qquad (k = 0, 1, 2, \cdots, 2p-1)$$

is unique.

§ 2

1. In this paragraph we examine complex sequences

$$c_0 \gtrless 0,\ c_1, c_2, \cdots, c_n \tag{1}$$

which are positive definite (or non-negative definite) on the entire circumference $0 \leq t \leq 2\pi$. For brevity we often simply call such sequences positive definite (or non-negative definite) without mention of the circumference.

By the theorem of Fejér and F. Riesz [35] every real trigonometric polynomial

$$T_n(e^{it}) = \sum_{k=-n}^{n} A_{-k} e^{ikt} \qquad (A_{-k} = \overline{A}_k),$$

which does not assume negative values may be expressed in the form[1)]

$$T_n(e^{it}) = \Big| \sum_{k=0}^{n} \xi_k e^{ikt} \Big|^2 = \sum_{0}^{n} \xi_\lambda \overline{\xi}_\mu e^{i(\lambda-\mu)t},$$

where ξ_k are some complex numbers. But then

$$\mathfrak{C}\{T_n\} = \sum_{0}^{n} c_{\lambda-\mu} \xi_\lambda \overline{\xi}_\mu.$$

From this is obtained

Theorem 6. *A sequence* (1) *is positive definite (or non-negative definite) if and only if the Toeplitz form*

$$\sum_{0}^{n} c_{\lambda-\mu} \xi_\lambda \overline{\xi}_\mu \tag{2}$$

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1) We shall indicate a proof of the theorem of Fejér and Kiesz. The quasi-polynomial

$$T_n(z) = \sum_{k=-n}^{n} A_k z^k$$

is symmetric, i.e.,

$$\overline{T}_n(z) = T_n\left(\frac{1}{z}\right).$$

Therefore, if ζ is a root of $T_n(z)$, then $\frac{1}{\zeta}$ is also a root; it follows from this that

$$T_n(z) = \frac{1}{z^m} \prod_{\lambda=1}^{k} (z - z_\lambda) \left(z - \frac{1}{\overline{z}_\lambda}\right) \prod_{\mu=1}^{l} (z - \zeta_\mu),$$

where $|z_\lambda| < 1$, $|\zeta_\mu| = 1$, $2k + l = 2m$, $m \leq n$ ($m < n$, if $A_n = 0$) or, written differently:

$$T_n(z) = \left\{ \prod_{\lambda=1}^{k} (z - z_\lambda) \left(\frac{1}{z} - \overline{z}_\lambda\right) \right\} \left\{ \frac{C'}{z^s} \prod_{\mu=1}^{2s} (z - \zeta_\mu) \right\},$$

where $s = m - k$.

The first factor has non-negative values for $z = e^{it}$; hence the second factor must have the same property and therefore, as is easily seen, cannot have roots of odd multiplicity, which means that we may set

$$\prod_{\mu=1}^{2s} (z - \zeta_\mu) = \prod_{\mu=1}^{s} (z - \zeta_\mu)^2.$$

Setting

$$\Gamma z^{n-m} \prod_{\lambda=1}^{k} (z - z_\lambda) \prod_{\mu=1}^{s} (z - \zeta_\mu) = \sum_{\nu=0}^{n} \xi_\nu z^\nu \quad (|\Gamma|^2 = C'),$$

we find that

$$T_n(z) = \sum_{\nu=0}^{n} \xi_\nu z^\nu \sum_{k=0}^{n} \overline{\xi}_k \frac{1}{z^k},$$

as was to be proved.

is positive (or non-negative).

If the sequence (1) is non-negative definite and not all of its elements are zero (from now on we consider only such sequences) then $c_0 > 0$. Thus, since (2) is a non-negative form,

$$c_0 \geqslant 0, \begin{vmatrix} c_0 & c_k \\ c_{-k} & c_0 \end{vmatrix} \geqslant 0 \quad \text{or} \quad c_0 \geqslant |c_k| \qquad (k = 0, 1, \ldots, n).$$

If the non-negative definite sequence $c_0 > 0,\ c_1, \cdots, c_n$ is not positive definite, then there exists an integer p $(1 \leq p \leq n)$ such that $c_0, c_1, \cdots, c_{p-1}$ is positive definite but the sequence $c_0, c_1, \cdots, c_p$ is not positive definite. We shall define the rank of the non-negative definite sequence $c_0, c_1, \cdots, c_n$ as the number p if this sequence is not positive definite and as the number $n + 1$ if it is positive definite.

Later we shall prove that the rank of the non-negative definite sequence (1) coincides with the rank of the Toeplitz form (2). In the meantime we may assert on the basis of Theorem 6 that in order for the sequence (1) to have rank p it is necessary and sufficient that

$$\Delta_k = \begin{vmatrix} c_0 & c_1 & \cdots & c_k \\ c_{-1} & c_0 & \cdots & c_{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{-k} & c_{-k+1} & \cdots & c_0 \end{vmatrix} > 0 \qquad (k = 0, 1, \ldots, n),$$

if $p = n + 1$, and that

$$\Delta_0 > 0,\ \Delta_1 > 0, \cdots, \Delta_{p-1} > 0,\ \Delta_p = 0,$$

if $p < n + 1$.

2. Let the sequence

$$c_0 > 0,\ c_1, c_2, \cdots, c_{n-1} \tag{3}$$

be positive definite.

We shall examine the region C_n in the complex plane ζ defined by the inequality

$$\Delta_n(\zeta) = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & \zeta \\ c_{-1} & c_0 & \cdots & c_{n-2} & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \overline{\zeta} & c_{-n+1} & \cdots & c_{-1} & c_0 \end{vmatrix} \geqslant 0. \tag{4}$$

We shall show that C_n is a circle with center at the point

$$\zeta_0 = \frac{(-1)^{n-1}}{\Delta_{n-2}} \begin{vmatrix} c_1 & c_2 & \cdots & c_{n-1} & 0 \\ c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{-n+2} & c_{-n+3} & \cdots & c_0 & c_1 \end{vmatrix} \tag{4_1}$$

and with radius

$$r = \frac{\Delta_{n-1}}{\Delta_{n-2}}. \tag{4_2}$$

In fact, on the basis of a well-known theorem about the minors of a reciprocal determinant

$$\Delta_{n-2}\Delta_n(\zeta) = \Delta_{n-1}^2 - \begin{vmatrix} c_1 & c_2 & \cdots & c_{n-1} & \zeta \\ c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{-n+2} & c_{-n+3} & \cdots & c_0 & c_1 \end{vmatrix} \begin{vmatrix} c_{-1} & c_0 & \cdots & c_{n-2} \\ c_{-2} & c_{-1} & \cdots & c_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{\zeta} & c_{-n+1} & \cdots & c_{-1} \end{vmatrix}.$$

Since $\Delta_{n-2} > 0$ and $\Delta_{n-1} > 0$, inequality (4) is then equivalent to the inequality

$$|\zeta - \zeta_0| \leq r.$$

Theorem 7. *Depending upon whether the point c_n lies on the boundary of the circle C_n, within it, or outside of it all roots of the polynomial*

$$W_n(z) = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{-n+1} & c_{-n+2} & \cdots & c_1 \\ 1 & z & \cdots & z^n \end{vmatrix} \tag{5}$$

lie on the boundary, inside or outside of the unit circle.

Proof. Setting

$$\mathfrak{C}\{z^n\} = c_n, \quad \mathfrak{C}\{z^{-n}\} = c_{-n} = \overline{c}_n,$$

we notice that $W_n(z)$ is characterized by the following condition of orthogonality:

$$\mathfrak{C}\{W_n(z)z^{-k}\} = 0 \qquad (k = 0, 1, \cdots, n-1). \tag{5_1}$$

Since

$$W_n(z) = \Delta_{n-1} z^n + \cdots,$$

we have

$$\mathfrak{C}\left\{W_n(z)\, W_n\left(\frac{1}{z}\right)\right\} = \mathfrak{C}\left\{W_n(z)\overline{W}_n\left(\frac{1}{z}\right)\right\} =$$
$$= \Delta_{n-1}\, \mathfrak{C}\{W_n(z)\, z^{-n}\} = \Delta_{n-1}\Delta_n(c_n). \qquad (5_2)$$

Let α now be a root of the polynomial $W_n(z)$. Then

$$W_n(z) = (z - \alpha)\, G(z),$$

where $G(z)$ is a polynomial of degree $n - 1$, so that

$$0 = \mathfrak{C}\left\{W_n(z)\,\overline{G}\left(\frac{1}{z}\right)\right\} = \mathfrak{C}\left\{(z-\alpha)\, G(z)\,\overline{G}\left(\frac{1}{z}\right)\right\} = B - \alpha A,$$

where

$$B = \mathfrak{C}\left\{zG(z)\,\overline{G}\left(\frac{1}{z}\right)\right\}, \quad A = \mathfrak{C}\left\{G(z)\,\overline{G}\left(\frac{1}{z}\right)\right\} > 0,$$

since the sequence (3) is positive definite.

Thus,

$$\alpha = \frac{B}{A}.$$

Noting that

$$\overline{B} = \mathfrak{C}\left\{\frac{1}{z}\, G(z)\,\overline{G}\left(\frac{1}{z}\right)\right\}$$

and taking (5_2) into consideration we find that

$$\Delta_{n-1}\Delta_n(c_n) = \mathfrak{C}\left\{(z-\alpha)\left(\frac{1}{z} - \overline{\alpha}\right) G(z)\,\overline{G}\left(\frac{1}{z}\right)\right\} =$$
$$= A(1 + |\alpha|^2) - \overline{\alpha}\, B - \alpha\overline{B} = A(1 - |\alpha|^2).$$

Thus, $\Delta_n(c_n)$ and $1 - |\alpha|^2$ are simultaneously > 0, < 0, or $= 0$, as was required to prove.

We notice that by Theorem 6 a sequence $c_0, c_1, \cdots, c_n$ is positive definite if c_n is chosen inside C_n, and non-negative definite (rank n) if c_n lies on the boundary of C_n.

In the last case we may indicate a more complete result than that given by Theorem 7.

Theorem 8. *If a sequence* $c_0, c_1, \cdots, c_{n-1}$ *is positive definite and* c_n *lies*

on the boundary of C_n, *i.e.*,

$$\Delta_n = \Delta_n(c_n) = |c_{\lambda-\mu}|_0^n = 0, \tag{6}$$

then all roots of the polynomial $W_n(z)$, *defined in* (5) *are distinct and have unit modulus.*

We give a proof which is independent of Theorem 7.

By (6) there exists a system of complex numbers $u_0, u_1, \cdots, u_n$ distinct from zero for which

$$\sum_{\lambda=0}^{n} c_{\lambda-\mu} u_\lambda = 0 \qquad (\mu = 0, 1, \cdots, n). \tag{6_1}$$

Replacing λ by $(n - \lambda)$ and μ by $(n - \mu)$ for this system of equations and passing to complex conjugates we also obtain

$$\sum_{\lambda=0}^{n} c_{\lambda-\mu} \bar{u}_{n-\lambda} = 0 \qquad (\mu = 0, 1, \cdots, n).$$

From this, since $\Delta_{n-1} > 0$:

$$\frac{\bar{u}_n}{u_0} = \frac{\bar{u}_{n-1}}{u_1} = \cdots = \frac{\bar{u}_1}{u_{n-1}} = \frac{\bar{u}_0}{u_n} = \epsilon \qquad (|\epsilon| = 1).$$

Since the system of numbers $u_0, u_1, \cdots, u_n$ is defined up to a multiplicative constant, the latter may be chosen so that $\epsilon = 1$.

In this case the polynomial $U_n(z) = u_0 + u_1 z + \cdots + u_n z^n$, which is obviously equal to $W_n(z)$ multiplied by a constant, is a symmetric polynomial

$$z^n U_n\left(\frac{1}{z}\right) = \bar{U}_n(z).$$

By (5_1), (5_2), and (6) or simply by (6_1)

$$\mathfrak{C}\{U_n(z) z^{-k}\} = 0 \qquad (k = 0, 1, 2, \cdots, n). \tag{7}$$

We denote by $e^{it_1}, e^{it_2}, \cdots, e^{it_m}$ all distinct roots of odd multiplicity of the polynomial $U_n(z)$, lying on the unit circle. Then we set

$$G(z) = e^{\frac{\pi i m}{2}} \cdot e^{\frac{i}{2}(t_1+t_2+\cdots+t_m)} (z - e^{-it_1}) \cdots (z - e^{-it_m}),$$

if $m \geq 1$, and $G(z) = 1$, if $m = 0$. It is easily seen that the polynomial $G(z)$ is symmetric.

Roots of the polynomial $U_n(z)$ not lying on the circle $|z| = 1$ must, if they

exist (and we do not rely upon Theorem 7), be symmetric with respect to the circle $|z| = 1$.

Therefore $n - m = 2l$ is an even number and $U_n(z)$ may be represented in the form

$$U_n(z) = \epsilon G(z) K(z) \overline{K}\left(\frac{1}{z}\right) z^l,$$

where $K(z)$ is a polynomial of degree l.

It is apparent from this that the quasi-polynomial

$$H(z) = \frac{\bar{\epsilon}}{z^l} \overline{G}\left(\frac{1}{z}\right) U_n(z)$$

for $z = e^{it}$ becomes a real trigonometric polynomial of order $m + l = n - l$, non-negative in the entire interval $0 \leq t \leq 2\pi$. Since the sequence (3) is positive definite for $l \geq 0$ we obtain

$$\mathfrak{C}\{H(z)\} > 0,$$

but by (7) this is absurd and hence $l = 0$, i.e., $m = n$, which completes the proof.

Theorem 9. *If $c_0 > 0, c_1, \cdots, c_{n-1}$ is a positive definite sequence, then to every complex number c_n lying on the circle C_n, whose equation is*

$$\begin{vmatrix} c_0 & c_1 \cdots c_{n-1} & \zeta \\ c_{-1} & c_0 \cdots c_{n-2} & c_{n-1} \\ \cdots & \cdots & \cdots \\ \bar{\zeta} & c_{-n+1} \cdots c_{-1} & c_0 \end{vmatrix} = 0,$$

there corresponds one and only one canonical representation

$$c_k = \sum_{j=1}^{n} \rho_j \alpha_j^k \qquad (k = 0, 1, \cdots, n-1), \tag{8}$$

in which $\rho_j > 0$, $|\alpha_j| = 1$ $(j = 1, 2, \cdots, n)$ and all the α_j are distinct and such that

$$c_n = \sum_{j=1}^{n} \rho_j \alpha_j^n. \tag{8'}$$

Conversely, to every representation (8) of the indicated type there corresponds by formula (8') some c_n lying on the circle C_n.

Proof. By Lagrange's formula an arbitrary polynomial $G(z)$ of degree $\leq n$ may be represented in the following form:

$$G(z) = \sum_{k=1}^{n} G(\alpha_k) h_k(z) + \text{const } U_n(z),$$

where α_k $(k = 1, 2, \cdots, n)$ are all (distinct) roots of $U_n(z)$ and

$$h_k(z) = \frac{U_n(z)}{(z-\alpha_k) U_n'(\alpha_k)}.$$

But since $\mathfrak{C}\{U_n\} = 0$,

$$\mathfrak{C}\{G\} = \sum_{k=1}^{n} \rho_k G(\alpha_k), \tag{9}$$

where

$$\rho_k = \mathfrak{C}\{h_k\}.$$

Setting $G(z) = z^l$ $(l = 0, 1, \cdots, n)$ in formula (9), we obtain representation (8), (8′).

It remains to show that $\rho_k > 0$ $(k = 1, 2, \cdots, n)$.

Noting that $h_k(\alpha_k) = 1$, $|\alpha_k| = 1$, we find that $\overline{h}_k(\frac{1}{\alpha_k}) = 1$ and thus

$$\overline{h}_k(\tfrac{1}{z}) - 1 = (z - \alpha_k) g_k(\tfrac{1}{z}),$$

where $g_k(z)$ is a polynomial of degree n. But then

$$h_k(z)\overline{h}_k\left(\frac{1}{z}\right) - h_k(z) = \frac{1}{U_n'(\alpha_k)} U_n(z) g_k\left(\frac{1}{z}\right).$$

Applying the functional $\mathfrak{C}$ on both sides of this equality, we get by the condition of orthogonality that

$$\rho_k = \mathfrak{C}\{h_k(z)\} = \mathfrak{C}\left\{h_k(z)\overline{h}_k\left(\frac{1}{z}\right)\right\} > 0,$$

since

$$h_k(e^{it})\overline{h}_k(e^{-it}) = |h_k(e^{it})|^2$$

is a non-negative trigonometric polynomial of order $n - 1$.

The uniqueness of the representation (8), (8′) may be proved in the following manner. It follows from (8) and (8′) that

$$\sum_{0}^{n-1} (c_{\lambda-\mu+1} - z c_{\lambda-\mu}) \xi_\lambda \overline{\xi}_\mu =$$

$$= \sum_{j=1}^{n} (\alpha_j - z) \rho_j |\xi_0 + \xi_1 \alpha_j + \ldots + \xi_{n-1}\alpha_j^{n-1}|^2.$$

The form written above is degenerate for $z = \alpha_j$ $(j = 1, 2, \cdots, n)$, therefore the numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are roots of the determinant

$$\delta_n(z) = \left| z c_{\lambda-\mu} - c_{\lambda-\mu+1} \right|_0^{n-1}.$$

This means that the system of numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ is determined uniquely and thus, $\delta_n(z) = \text{const } U_n(z)$; moreover, it is immediately seen that $\delta_n(z) = U_n(z)$.

Once the numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are determined, the numbers $\rho_1, \rho_2, \cdots, \rho_n$ are uniquely determined from equations (8).

For the proof of the second part of the theorem we note that from (8) and (8′) also follows

$$\sum_0^n c_{\lambda-\mu} \xi_\lambda \bar{\xi}_\mu = \sum_{j=1}^n \rho_j \left| \xi_0 + \xi_1 \alpha_j + \ldots + \xi_n \alpha_j^n \right|^2.$$

The form above is degenerate and thus

$$\left| c_{\lambda-\mu} \right|_0^n = 0.$$

This proves the theorem completely.

Theorem 10. *If*

$$c_k = \sum_{j=1}^n \rho_j \alpha_j^k \qquad (k = 0, 1, \cdots, n-1) \tag{8}$$

and

$$c_k = \sum_{j=1}^n \rho_j' \alpha_j'^k \qquad (k = 0, 1, \cdots, n-1) \tag{8_1}$$

are the two distinct canonical representations discussed in the previous theorem, then the points $\alpha_1, \alpha_2, \cdots, \alpha_n$ *and* $\alpha_1', \alpha_2', \cdots, \alpha_n'$ *alternate on the unit circle.*

Proof. Let

where

$$\alpha_k = e^{it_k}, \quad \alpha_k' = e^{it_k'} \qquad (k = 1, 2, \ldots, n),$$
$$0 \leqslant t_1 < t_2 < \ldots < t_n < 2\pi,$$
$$0 \leqslant t_1' < t_2' < \ldots < t_n' < 2\pi.$$

We must show that within any of the intervals (t_k, t_{k+1}) $(k = 1, 2, \cdots, n;\ t_{n+1} = t_1 + 2\pi)$ there lies at least one of the points t_j' $(j = 1, 2, \cdots, n+1;\ t_{n+1}' = t_1' + 2\pi)$

We shall show this for the interval (t_1, t_2) as an example. For this purpose we introduce for consideration the quasi-polynomial

$$\Phi(z) = \text{const}\,\frac{1}{z^{n-1}}(z-e^{it_1})(z-e^{it_2})\prod_{k=3}^{n}(z-e^{it_k})^2,$$

which may be alternatively defined by the equation

$$\Phi(e^{it}) = \sin\frac{t-t_1}{2}\sin\frac{t-t_2}{2}\prod_{k=3}^{n}\sin^2\frac{t-t_k}{2}.$$

Obviously

$$\begin{aligned}\Phi(e^{it}) &> 0 \quad \text{for} \quad t_2 < t < t_1 + 2\pi,\\ \Phi(e^{it}) &< 0 \quad \text{for} \quad t_1 < t < t_2.\end{aligned} \tag{10}$$

Because $\Phi(\alpha_k) = 0$, for $k = 1, 2, \cdots, n$, it then follows from (8) that

$$\mathfrak{S}\{\Phi\} = \sum_{k=1}^{n}\rho_k\Phi(\alpha_k) = 0,$$

from which by (8_1)

$$\mathfrak{S}\{\Phi\} = \sum_{k=1}^{n}\rho_k'\Phi(\alpha_k') = 0.$$

Because the systems of numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\alpha_1', \alpha_2', \cdots, \alpha_n'$ are distinct, not all the $\Phi(\alpha_k')$ are equal to zero and thus among the numbers $\Phi(\alpha_k')$ $(k = 1, 2, \cdots, n)$ there exist numbers with different signs. By (10) this means that in each of the intervals (t_1, t_2) and $(t_2, t_1 + 2\pi)$ there occur points t_k' $(k = 1, 2, \cdots, n+1)$, which completes the proof.

Theorem 11. *If the point c_n undergoes a continuous, clockwise motion around the circle C_n, then the points $\alpha_1, \alpha_2, \cdots, \alpha_n$ corresponding to the point c_n in the canonical representations* (8), (8′) *move continuously in the clockwise direction on the circle $|z| = 1$. When the point c_n makes a complete revolution of C_n, the points $\alpha_1, \alpha_2, \cdots, \alpha_n$ are permuted cyclically.*

Proof. Because the points $\alpha_1, \alpha_2, \cdots, \alpha_n$ are roots of the polynomial $W_n(z)$, whose coefficients are linear in c_n, the $\alpha_k = \alpha_k(c_n)$ $(k = 1, 2, \cdots, n)$ are analytic functions of c_n. On the other hand, since to two different values of c_n there correspond alternating systems of numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$, which consequently have no common elements, all the points α_k move in the same unchanging direction if c_n moves in one unchanging direction around C_n. Moreover, if c_n makes a complete revolution the points $\alpha_1, \alpha_2, \cdots, \alpha_n$ will be permuted, and obviously the only possible order of permutation is cyclical.

It remains to prove that to a clockwise motion of c_n there corresponds a clockwise movement of the points $\alpha_1, \alpha_2, \cdots, \alpha_n$.

For this purpose note that if a point c_n goes from the circle C_n into its interior, then according to Theorem 7 the points α_k go from the boundary of the unit circle into its interior. Since the mapping realized by the function $\alpha_k(c_n)$ is conformal, the correspondence of the directions of movement follows.[1)]

3. Let us examine the case in which the sequence (3) is non-negative definite and has rank $p < n$. In this case the sequence $c_0, c_1, \cdots, c_{p-1}$ is positive definite and

$$\Delta_p = \begin{vmatrix} c_0 & c_1 \cdots & c_p \\ c_{-1} & c_0 \cdots & c_{p-1} \\ \cdots & \cdots & \cdots \\ c_{-p} & c_{-p+1} \cdots & c_0 \end{vmatrix} = 0.$$

Thus by Theorem 9 the sequence $c_0, c_1, \cdots, c_p$ admits a canonical representation

$$c_k = \sum_{j=1}^{p} \rho_j \alpha_j^k \qquad (k = 0, \pm 1, \cdots, \pm p),$$

where all ρ_j $(j = 1, \cdots, p)$ are positive and $\alpha_1, \alpha_2, \cdots, \alpha_p$ are roots of the polynomial

$$U_p(z) = \text{const} \begin{vmatrix} c_0 & c_1 \cdots & c_p \\ c_{-1} & c_0 \cdots & c_{p-1} \\ \cdots & \cdots & \cdots \\ 1 & z \cdots & z^p \end{vmatrix} = u_0 + u_1 z + \ldots + u_p z^p \quad (|u_0| = |u_p| \neq 0)$$

(the numbers α_j are all distinct and of unit modulus).

From the orthogonality conditions

$$\mathfrak{C}\{U_p(z) z^{-k}\} = 0 \qquad (k = 0, 1, \cdots, p) \tag{11}$$

we have

$$\mathfrak{C}\{U_p(z) \overline{U}_p(\tfrac{1}{z})\} = 0. \tag{11_1}$$

Let us consider the quasi-polynomials

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1) This theorem could be proved by purely algebraic methods but the proof would be somewhat longer.

$$\Phi_k(z) = (A + Bz^k)\left(\overline{A} + \frac{\overline{B}}{z^k}\right) U_p(z)\,\overline{U}_p\left(\frac{1}{z}\right) \quad (k = 1,\ 2, \ldots,\ n - p - 1),$$

to each of which there corresponds a non-negative trigonometric polynomial $\Phi_k(e^{it})$; thus,

$$\mathfrak{C}\{\Phi_k\} \geq 0 \qquad (k = 1,\ 2,\ \cdots,\ n - p - 1).$$

On the other hand, if we consider (11_1) we obtain

$$0 \leqq \mathfrak{C}\{\Phi_k\} = \overline{A}B\,\mathfrak{C}\left\{z^k\, U_p(z)\,\overline{U}_p\left(\frac{1}{z}\right)\right\} + A\overline{B}\,\mathfrak{C}\left\{z^{-k} U_p(z)\,\overline{U}_p\left(\frac{1}{z}\right)\right\}.$$

In view of the complete arbitrariness of the constants A and B it follows from this inequality that

$$\mathfrak{C}\left\{U_p(z)\,\overline{U}_p\left(\frac{1}{z}\right) z^k\right\} = 0 \qquad [k = \pm 1,\ \pm 2, \ldots,\ \pm(n - p - 1)]. \quad (11_2)$$

Since

$$\overline{U}_p\left(\frac{1}{z}\right) = \overline{u}_0 + \overline{u}_1 \frac{1}{z} + \ \cdots\ + \overline{u}_p \frac{1}{z^p},$$

the relationship (11_2) may be written:

$$\overline{u}_0\,\mathfrak{C}\{U_p(z)\, z^k\} + \overline{u}_1\,\mathfrak{C}\{U_p(z)\, z^{k-1}\} + \ \ldots\ + \overline{u}_p\,\mathfrak{C}\{U_p(z)\, z^{k-p}\} = 0.$$

Setting $k = 1,\ 2,\ \cdots,\ n - p - 1$ successively and using (11), we get

$$\mathfrak{C}\{U_p(z)\, z^k\} = 0 \qquad (k = 0,\ 1,\ 2,\ \cdots,\ n - p - 1). \qquad (11')$$

Now let $G(z) = A_0 + A_1 z + \cdots + A_{n-1} z^{n-1}$ be any polynomial of degree $n - 1$; we represent it in the form

$$G(z) = U_p(z)\, q(z) + r(z),$$

where $r(z)$ is some polynomial of degree $\leq p - 1$. By (11'):

$$\mathfrak{C}\{G\} = \mathfrak{C}\{r\}.$$

On the other hand

$$\mathfrak{C}\{r\} = \sum_{k=1}^{p} \rho_k\, r(\alpha_k) = \sum_{k=1}^{p} \rho_k\, G(\alpha_k),$$

since

$$r(\alpha_k) = G(\alpha_k) \qquad (k = 1,\ 2,\ \cdots,\ p).$$

Therefore for any polynomial $G(z)$ of degree $\leq n - 1$ one has the "quadrature formula"

$$\mathfrak{S}\{G\} = \sum_{k=1}^{p} \rho_k G(\alpha_k). \tag{12}$$

Hence

Theorem 12. *If a non-negative definite sequence* $c_0, c_1, \cdots, c_{n-1}$ *has rank* $p < n$, *then there exists a unique canonical representation*

$$c_k = \sum_{j=1}^{q} \rho_j \alpha_j^k \qquad [k = 0, \pm 1, \cdots, \pm (n-1)], \tag{13}$$

in which $\rho_j > 0$, $|\alpha_j| = 1$ *and all* α_j *are distinct. In this representation* $q = p =$ *rank of the Toeplitz form*

$$\sum_{0}^{n-1} c_{\lambda-\mu} \xi_\lambda \bar{\xi}_\mu. \tag{14}$$

Proof. Setting $G(z) = z^k$ in (12) and taking complex conjugates we in fact obtain representation (13) with $q = p$. On the other hand, given a canonical representation (13), we can expand form (14) into squares, namely:

$$\sum_{0}^{n-1} c_{\lambda-\mu} \xi_\lambda \bar{\xi}_\mu = \sum_{j=1}^{q} \rho_j |\xi_0 + \xi_1 \alpha_j + \cdots + \xi_{n-1} \alpha_j^{n-1}|^2.$$

It follows from this expansion that $q < n$ since otherwise this form would be positive and thus the sequence $c_0, c_1, \cdots, c_{n-1}$ would be positive definite.

But if $q < n$, then q is equal to the rank of form (14) and is thus uniquely defined; therefore $q = p$.

On the other hand if $q = p$, the $2p$ numbers ρ_j and α_j $(j = 1, 2, \cdots, p)$ are, by Theorem 9, uniquely determined by the equations

$$c_k = \sum_{j=1}^{p} \rho_j \alpha_j^k \qquad (k = 0, 1, \cdots, p).$$

§3

1. Let us concentrate on the case in which the sequence

$$c_0 > 0, c_1, c_2, \cdots, c_{n-1} \qquad (c_k = \bar{c}_k;\ k = 0, 1, \cdots, n-1) \tag{1}$$

is real.

We shall assume initially that the sequence (1) is positive definite. According to formula (4_1), §2 the center of the circle C_n lies on the real axis and therefore

by (4_1), (4_2) §2 the circle C_n intersects this axis at the two points

$$\zeta_{1,2} = \frac{(-1)^{n-1}}{\Delta_{n-2}} \begin{vmatrix} c_1 & c_2 \cdots & c_{n-1} & 0 \\ c_0 & c_1 \cdots & c_{n-2} & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{-n+2} & c_{-n+3} \cdots & c_0 & c_1 \end{vmatrix} \mp \frac{\Delta_{n-1}}{\Delta_{n-2}}. \tag{2}$$

Hence, if the sequence (1) is positive definite, in order that the extended sequence $c_0, c_1, \cdots, c_n$ with real c_n be non-negative definite it is necessary and sufficient that the number c_n be located between the limits $\zeta_1 < \zeta_2$, which are *rationally expressed in terms of the* $c_0, c_1, \cdots, c_{n-1}$ by formula (2).

Let us now consider the canonical representation

$$c_k = \sum_{j=1}^{n} \rho_j \alpha_j^k \quad (k = 0, 1, \cdots, n) \tag{3}$$

corresponding, by Theorem 9, to some real value of c_n lying on the circumference of C_n.

Taking complex conjugates in representation (3) we have, by virtue of the fact that the c_k $(k = 0, 1, \cdots, n)$ are real, that

$$c_k = \sum_{j=1}^{n} \rho_j \bar{\alpha}_j^k.$$

Since by Theorem 9 there exists only one canonical representation (3), the system $\rho_j \alpha_j^k$ $(j = 1, \cdots, n)$ coincides with the system $\rho_j \bar{\alpha}_j^k$ $(j = 1, \cdots, n)$.

From this it follows that for even n either all the numbers α_j are different from -1 and $+1$, or else -1 and $+1$ simultaneously belong to the system $\alpha_1, \alpha_2, \cdots, \alpha_n$; on the other hand, for odd n one of the numbers ± 1 must belong to the system $\alpha_1, \alpha_2, \cdots, \alpha_n$.

We shall show that each of these possibilities is realized for one or the other choice of c_n from the two values ζ_1, ζ_2.

In fact, as soon as c_n is chosen, the numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ in representation (3) are determined as roots of the polynomial

$$W_n(z) = \text{const}\, U_n(z) = \begin{vmatrix} c_0 c_1 \cdots c_n \\ c_1 c_0 \cdots c_{n-1} \\ \cdots \cdots \cdots \\ 1 \; z \cdots z^n \end{vmatrix} = w_0 + w_1 z + \cdots + w_n z^n,$$

where by the symmetry of the polynomial $U_n(z)$:

$$\frac{w_0}{w_n} = \frac{w_1}{w_{n-1}} = \cdots = \frac{w_n}{w_0} = \pm 1. \tag{4}$$

On the other hand if we substitute c_n for $\zeta_{1,2}$ in (2), then after small changes it is possible to reduce equation (2) to the form

$$w_0 - w_n = 0$$

for $c_n = \zeta_1$ and to the form

$$w_0 + w_n = 0$$

for $c_n = \zeta_2$.

From this, taking into consideration (4), we find that if $n = 2m$, then

$$W_n(-1) \neq 0, \quad W_n(1) \neq 0 \quad \text{for} \quad c_n = \zeta_1$$

and

$$W_n(-1) = W_n(1) = 0 \quad \text{for} \quad c_n = \zeta_2;$$

but if $n = 2m - 1$, then

$$W_n(-1) = 0, \quad W_n(1) \neq 0 \quad \text{for} \quad c_n = \zeta_1$$

and

$$W_n(-1) \neq 0, \quad W_n(1) = 0 \quad \text{for} \quad c_n = \zeta_2.$$

Noting on the other hand that in representation (3)

$$\rho_j \alpha_j^k + \rho_j \bar{\alpha}_j^k = 2\rho_j \cos kt_j \qquad (\alpha_j = e^{it_j}),$$

we obtain the theorem.

Theorem 13. *If the real sequence*

$$c_0, c_1, \cdots, c_{2m-1}$$

is positive definite there exists a canonical representation

$$c_k = \sum_{j=1}^{m} \rho_j \cos kt_j \qquad (k = 0, 1, \cdots, 2m-1), \tag{5}$$

where

$$\rho_j > 0 \qquad (j = 1, 2, \cdots, m)$$

and

$$0 < t_1 < t_2 < \cdots < t_m < \pi, \tag{5$_1$}$$

and also a canonical representation

$$c_k = \sum_{j=0}^{m} \rho_j' \cos kt_j' \qquad (k = 0, 1, \cdots, 2m-1),$$

where

$$\rho_j' > 0 \qquad (j = 0, 1, \cdots, m)$$

and

$$0 = t_0' < t_1' < \cdots < t_m' = \pi;$$

and moreover

$$t_0' < t_1 < t_1' < \cdots < t_m < t_m'.$$

Theorem 14. *If a real sequence*

$$c_0, c_1, \cdots, c_{2m-2}$$

is positive definite, then there exists a canonical representation

$$c_k = \sum_{j=1}^{m} \rho_j \cos kt_j \qquad (k = 0, 1, \cdots, 2m-2),$$

where

$$\rho_j > 0 \qquad (j = 1, 2, \cdots, m)$$

and

$$0 < t_1 < t_2 < \cdots < t_m = \pi,$$

and also a canonical representation

$$c_k = \sum_{j=1}^{m} \rho_j' \cos kt_j' \qquad (k = 0, 1, \cdots, 2m-2),$$

where

$$\rho_j' > 0 \qquad (j = 1, 2, \cdots, m)$$

and

$$0 = t_1' < t_2' < \cdots < t_m' < \pi;$$

and moreover

$$t_1' < t_1 < t_2' < t_2 < \cdots < t_m' < t_m.$$

The alternation of the numbers t_j, t_j' in both theorems follows from the general Theorem 10. The last theorem may be supplemented in the following manner:

Theorem 15. *If a real sequence*

$$c_0, c_1, \cdots, c_{2m-2}$$

is positive definite, there exist infinitely many canonical representations

$$c_k = \sum_{j=1}^{m} \rho_j \cos kt_j \qquad (k = 0, 1, \cdots, 2m-2), \tag{6}$$

where

$$\rho_j > 0 \qquad (j = 1, 2, \cdots, m)$$

and

$$0 \le t_1 < t_2 < \cdots < t_m \le \pi; \tag{6_1}$$

the systems $t_1, t_2, \cdots, t_m$ *corresponding to two distinct representations alternate.*

Proof. As we know, there exist infinitely many values $c_n = c_{2m-1}$ for which the sequence

$$c_0, c_1, \cdots, c_{2m-2}, c_{2m-1}$$

is positive definite and to every such sequence there corresponds some canonical representation (5), (5_1); and this implies (6), (6_1).

The strict alternation follows from ideas analogous to those used in proving Theorems 3 and 10.

Let

$$c_k = \sum_{j=1}^{m} \rho_j' \cos kt_j' \qquad (k = 0, 1, \cdots, 2m-2) \tag{$6'$}$$

be one of the representations of the type (6), (6_1). We define the polynomial $\Phi(x)$ of degree $2m-2$ by the equation

$$\Phi(x) = \frac{1}{(x - \cos t_k')(x - \cos t_{k+1}')} \prod_{1}^{m} (x - \cos t_j')^2.$$

From ($6'$):

$$\mathfrak{C}\left\{\Phi\left(\frac{z+z^{-1}}{2}\right)\right\} = \sum_{j=1}^{m} \rho_j' \Phi(\cos t_j') = 0.$$

Therefore, for every representation (6), (6_1)

$$\mathfrak{C}\left\{\Phi\left(\frac{z+z^{-1}}{2}\right)\right\} = \sum_{j=1}^{m} \rho_j \Phi(\cos t_j) = 0. \tag{7}$$

In view of the fact that

$$\Phi(\cos t) < 0,$$

if $t_k' < t < t_{k+1}'$ and

$$\Phi(\cos t) > 0,$$

if t lies outside $< t_k', t_{k+1}' >$ and $0 \leq t \leq \pi$, then it follows from (7) that at least one of the t_j falls within the interval (t_k', t_{k+1}') if the system $t_1 < t_2 < \cdots < t_m$ does not coincide with the system $t_1' < t_2' < \cdots < t_m'$. Q. E. D.

2. We shall now substitute the condition that the sequence be non-negative definite for the condition that it be positive definite.

Theorem 16. *If a real non-negative definite sequence*

$$c_0 > 0, c_1, \cdots, c_{n-1}$$

has rank $p < n$, then there exists a unique canonical representation

$$c_k = \sum_{j=1}^{m} \rho_j \cos kt_j \quad (k = 0, 1, \cdots, n-1), \tag{8}$$

where

$$\rho_j > 0 \quad (j = 1, 2, \cdots, m)$$

and

$$0 \leq t_1 < t_2 < \cdots < t_m \leq \pi.$$

Moreover, if we put

$$\epsilon_j = \begin{cases} 2 \text{ for } 0 < t_j < \pi \\ 1 \text{ for } t_j = 0 \text{ or } \pi, \end{cases}$$

then

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m = p.$$

In fact on the basis of Theorem 12 there exists a unique canonical representation

$$c_k = \sum_{j=1}^{p} \rho_j \alpha_j^k \qquad (k = 0, 1, 2, \cdots, n-1), \tag{8'}$$

where $\rho_j > 0$, $|\alpha_j| = 1$ $(j = 1, \cdots, p)$ and α_j are all distinct. From this, since $c_0, c_1, \cdots, c_{n-1}$ are real, the system of numbers $\rho_j \alpha_j^k$ $(j = 1, 2, \cdots, p)$ coincides with the system $\rho_j \bar{\alpha}_j^k$ $(j = 1, 2, \cdots, p)$. Combining the conjugate members of (8'), we arrive at the representation (8).

Since

$$\Delta_p = \left| c_{\lambda-\mu} \right|_0^p = 0,$$

we have

$$c_p = \frac{(-1)^{p-1}}{\Delta_{p-2}} \begin{vmatrix} c_1 & c_2 \dots & c_{p-1} & 0 \\ c_0 & c_1 \dots & c_{p-2} & c_{p-1} \\ \dots & \dots & \dots & \dots \\ c_{p-2} & c_{p-3} \dots & c_0 & c_1 \end{vmatrix} \mp \frac{\Delta_{p-1}}{\Delta_{p-2}}.$$

Depending upon whether the + or − is appropriate and whether p is even or odd in representation (8), by Theorems 13, 14 one of four possible cases holds:

1. $t_1 > 0$, $t_m < \pi$; $2m = p$;
2. $t_1 = 0$, $t_m = \pi$; $2m - 2 = p$;
3. $t_1 > 0$, $t_m = \pi$; $2m - 1 = p$;
4. $t_1 = 0$, $t_m < \pi$; $2m - 1 = p$.

§4

1. In the present paragraph we shall show how the real positive definite (non-negative definite) sequence

$$c_0, c_1, \dots, c_{n-1}, c_n \tag{1}$$

on the circumference may be transformed into the sequence

$$s_0, s_1, \dots, s_{n-1}, s_n \tag{2}$$

which is positive definite (non-negative definite) in a finite interval which without loss of generality we may take as $<-1, 1>$.

In this fashion we shall obtain a series of propositions relating to the positive definite (non-negative definite) sequences in an interval.

All of these propositions may be obtained, however, by the methods expounded in §1.

We note in passing that theorems relating to sequences which are positive definite (non-negative definite) in the interval $(-\infty, \infty)$ may be obtained, as E. Fischer [25] showed, by proceeding from the theorems about positive definite (non-negative definite) sequences on the circumference.

We shall agree to define the rank of a sequence (2) non-negative definite in the interval $<-1, 1>$ as the greatest number p $(p \leq n + 1)$ satisfying the condition that the sequence

$$s_0, s_1, \cdots, s_{p-1}$$

is positive definite in the interval $< -1, 1 >$.

Theorem 17. *A real sequence*

$$c_0, c_1, \cdots, c_n \tag{1}$$

is non-negative definite on the circumference and has rank p $(p \leq n+1)$ *if and only if the sequence*

$$s_0, s_1, \cdots, s_n \tag{2}$$

where

$$s_k = \frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} c_{k-2r} \qquad (k = 0, 1, \cdots, n;\ c_{-k} = c_k) \tag{3}$$

is non-negative definite in the interval $< -1, 1 >$ *and has rank* p.

Proof. Equation (3) may be written in the following form:

$$s_k = \mathfrak{C}\left\{\frac{1}{2^k}\left(z + \frac{1}{z}\right)^k\right\} \qquad (k=0, 1, \ldots, n). \tag{3'}$$

Therefore for any polynomial $G(u) = \sum_{k=0}^{n} A_k u^k$:

$$\mathfrak{S}\{G(u)\} = \mathfrak{C}\left\{G\left(\frac{z+z^{-1}}{2}\right)\right\}. \tag{3''}$$

On the other hand if

$$G(u) \geq 0 \qquad \text{for } -1 \leq u \leq 1, \tag{4}$$

then

$$G\left(\frac{z+z^{-1}}{2}\right) = G(\cos t) \geqslant 0 \qquad (z = e^{it}).$$

Therefore if the sequence (1) is positive definite (non-negative definite) on the circumference, then by (3″) the inequality (4) implies the inequality

$$\mathfrak{S}\{G\} > 0 \qquad (\text{or } \mathfrak{S}\{G\} \geq 0),$$

i.e., the sequence (2) is positive definite (non-negative definite) in the interval $< -1, 1 >$.

On the other hand, suppose it is given that the sequence (2) is positive definite (non-negative definite) in the interval $< -1, 1 >$. To prove that the sequence (1) is positive definite (non-negative definite) it is necessary to demonstrate that the Toeplitz form

$$\sum_{0}^{n} c_{\lambda-\mu} \xi_\lambda \overline{\xi}_\mu$$

is positive (non-negative). But because the c_k are real numbers, in place of this form it is sufficient to examine the form

$$\sum^{n} c_{\lambda-\mu}\,\xi_\lambda\,\xi_\mu = \mathfrak{C}\{(\xi_0+\xi_1 z+\ldots+\xi_n z^n)(\xi_0+\xi_1 z^{-1}+\ldots+\xi_n z^{-n})\}$$

with real ξ_k.

On the other hand, since $(z = e^{it})$

$$T_n(z) = (\xi_0+\xi_1 z+\ldots+\xi_n z^n)(\xi_0+\xi_1 z^{-1}+\ldots+\xi_n z^{-n}) = \left|\sum_{k=0}^{n}\xi_k e^{ikt}\right|^2 = \left|\sum_{k=0}^{n}\xi_k e^{i\left(\frac{n}{2}-k\right)t}\right|^2, \tag{5}$$

we have

$$T_{2m}(z) = \left[\sum_{k=0}^{m} u_{m-k}\cos kt\right]^2 + \sin^2 t\left[\sum_{k=1}^{m} u_{m+k}\frac{\sin kt}{\sin t}\right]^2,$$

where

$$u_m = \xi_m,\quad u_{m-k} = \xi_{m-k}+\xi_{m+k},\quad u_{m+k} = \xi_{m-k}-\xi_{m+k} \qquad (k = 1, 2, \ldots m)$$

and

$$T_{2m-1}(z) = \cos^2\frac{t}{2}\left[\sum_{k=1}^{m} v_{m-k}\frac{\cos\left(k-\frac{1}{2}\right)t}{\cos\frac{1}{2}t}\right]^2 +$$

$$+\sin^2\frac{t}{2}\left[\sum_{k=1}^{m} v_{m+k}\frac{\sin\left(k-\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right]^2,$$

with

$$v_{m-k} = \xi_{m-k}+\xi_{m+k-1},\quad v_{m+k} = \xi_{m-k}-\xi_{m+k-1} \qquad (k = 1, 2, \ldots, m).$$

Introducing new variables X_λ, Y_λ by the equations

$$\sum_{k=0}^{m} u_{m-k}\cos kt = \sum_{k=0}^{m} X_k\cos^k t,$$

$$\sum_{k=1}^{m} u_{m+k}\frac{\sin kt}{\sin t} = \sum_{k=0}^{m-1} Y_k\cos^k t$$

for $n = 2m$ and the equations

$$\sum_{k=1}^{m} v_{m-k} \frac{\cos\left(k-\frac{1}{2}\right)t}{\cos\frac{1}{2}t} = \sum_{k=0}^{m-1} X_k \cos^k t,$$

$$\sum_{k=1}^{m} v_{m+k} \frac{\sin\left(k-\frac{1}{2}\right)t}{\sin\frac{1}{2}t} = \sum_{k=0}^{m-1} Y_k \cos^k t$$

for $n = 2m-1$, we find

$$T_{2m}(z) = \left(\sum_{k=0}^{m} X_k u^k\right)^2 + (1-u^2)\left(\sum_{k=0}^{m-1} Y_k u^k\right)^2,$$

$$T_{2m-1}(z) = (1+u)\left(\sum_{k=0}^{m-1} X_k u^k\right)^2 + (1-u)\left(\sum_{k=0}^{m-1} Y_k u^k\right)^2,$$

where $u = \frac{1}{2}\left(z+\frac{1}{z}\right) = \cos t$.

Therefore

$$\mathfrak{C}\{T_{2m}(z)\} = \mathfrak{S}\left\{\left(\sum_{k=0}^{m} X_k u^k\right)^2\right\} + \mathfrak{S}\left\{(1-u^2)\left(\sum_{k=0}^{m-1} Y_k u^k\right)^2\right\} \geqslant 0,$$

$$\mathfrak{C}\{T_{2m-1}(z)\} = \mathfrak{S}\left\{(1+u)\left(\sum_{k=0}^{m-1} X_k u^k\right)^2\right\} + \mathfrak{S}\left\{(1-u)\left(\sum_{k=0}^{m-1} Y_k u^k\right)^2\right\} \geqslant 0, \tag{5$_1$}$$

since each of the polynomials under the symbol $\mathfrak{S}$ is non-negative in the interval $<-1, 1>$. The theorem is proved.

Remark. We note that because of relations (3) the following formal identity holds:

$$\sqrt{x^2-1}\left\{\frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \ldots\right\} = c_0 + 2c_1 w + 2c_2 w^2 + \ldots \quad \left(x = \frac{1+w^2}{2w}\right), \tag{6}$$

where $\sqrt{x^2-1} > 0$ for $x > 1$ $(0 < w < 1)$.

We may deduce this relation with the help of (3′) assuming its generality is not destroyed if the sequences $\{c_k\}$, $\{s_k\}$ are infinite.

We have formally

$$\sqrt{x^2-1}\left\{\frac{s_0}{x}+\frac{s_1}{x^2}+\frac{s_2}{x^3}+\ldots\right\}=\sqrt{x^2-1}\,\mathfrak{C}\left\{\sum_{k=0}^{\infty}\frac{1}{2^k}\left(z+\frac{1}{z}\right)^k\frac{1}{x^{k+1}}\right\}=$$

$$=\sqrt{x^2-1}\,\mathfrak{C}\left\{\frac{1}{x-\frac{1}{2}\left(z+\frac{1}{z}\right)}\right\}=(1-w^2)\,\mathfrak{C}\left\{\frac{z}{(z-w)(1-wz)}\right\}=$$

$$=\mathfrak{C}\left\{\frac{w}{z-w}\right\}+\mathfrak{C}\left\{\frac{1}{1-wz}\right\}=\mathfrak{C}\left\{\sum_{k=0}^{\infty}\frac{w^{k+1}}{z^{k+1}}\right\}+\mathfrak{C}\left\{\sum_{k=0}^{\infty}w^kz^k\right\}=$$

$$=c_0+2c_1w+2c_2w^2+\ldots$$

Note also that formulas (3′) may be compared with the formula

$$c_k=\mathfrak{S}\{H_k(u)\}\qquad(k=0,\,1,\,\cdots,\,n),\tag{3‴}$$

where $H_k(u)$ is a polynomial in u of degree k, which is an inversion of formula (3).

Indeed, if we set

$$\frac{1}{2}(z^k+z^{-k})=H_k\left(\frac{z+z^{-1}}{2}\right)\qquad(k=0,\,1,\,\cdots),$$

then by (3″)

$$\mathfrak{S}\{H_k(u)\}=\mathfrak{C}\left\{\frac{z^k+z^{-k}}{2}\right\},$$

and on the other hand

$$\mathfrak{C}\left\{\frac{z^k+z^{-k}}{2}\right\}=\frac{1}{2}(c_k+c_{-k})=c_k\qquad(k=0,\,1,\,\cdots).$$

For the derivation of relations (5_1) (for quasi polynomial (5)) we did not depend on any special properties of the sequence $c_0, c_1, \cdots, c_n$ or of the transformed sequence $s_0, s_1, \cdots, s_n$. Therefore, using the distributive property of the functionals $\mathfrak{C}$ and $\mathfrak{S}$ and replacing $\mathfrak{C}\{z^k\}$ by c_k $(k=0,\pm1,\cdots,\pm n)$ and $\mathfrak{S}\{u^k\}$ by s_k $(k=0,\,1,\,\cdots,\,n)$ in (5_1) we obtain the following proposition.

Theorem 18. *A real Toeplitz form*

$$\sum_0^n c_{\lambda-\mu}\xi_\lambda\xi_\mu$$

may be transformed into the sum of two independent forms for $n=2m$ as follows:

$$\sum_0^m s_{i+k}X_iX_k+\sum_0^{m-1}(s_{i+k}-s_{i+k+2})\,Y_iY_k,\tag{7$_1$}$$

and for $n=2m-1$ into the sum of two independent forms:

$$\sum_{0}^{m-1}(s_{l+k}+s_{l+k+1})X_lX_k+\sum_{0}^{m-1}(s_{l+k}-s_{l+k+1})Y_lY_k; \qquad (7_2)$$

where the numbers c_k, s_k $(k = 0, 1, \cdots, n)$ *are connected by relation* (6). *The converse of this theorem is valid also.*

By Theorems 12, 17 the following corollary of Theorem 18 readily follows.

Corollary. *The rank of a sequence* (2) *which is non-negative definite in the interval* $<-1, 1>$ *is equal to the sum of the ranks of the two quadratic forms* (7_1) *for* $n = 2m$ *and forms* (7_2) *for* $n = 2m-1$.

2. Comparing Theorems 6, 17, 18 we are led to the proposition:

Theorem 19. *In order that a sequence* (2) *be positive definite (non-negative definite) in the interval* $<-1, 1>$ *it is necessary and sufficient that the two quadratic forms*

$$\sum_{0}^{m}s_{l+k}X_lX_k, \qquad \sum_{0}^{m-1}(s_{l+k}-s_{l+k+2})X_lX_k$$

for $n = 2m$ *and the two quadratic forms*

$$\sum_{0}^{m-1}(s_{l+k}+s_{l+k+1})X_lX_k, \qquad \sum_{0}^{m-1}(s_{l+k}-s_{l+k+1})X_lX_k$$

for $n = 2m-1$ *be positive (or non-negative).*

This theorem also follows from a proposition of F. Lukacs [35] according to which every polynomial $G_n(u)$ of degree $\leq n$ which satisfies the condition

$$G_n(u) \not\equiv 0, \quad G_n(u) \geq 0 \quad \text{for } -1 \leq u \leq 1$$

admits a representation

$$G_n(u)=\left(\sum_{k=0}^{m}X_ku^k\right)^2+(1-u^2)\left(\sum_{k=0}^{m-1}Y_ku^k\right)^2 \qquad \text{for} \qquad n=2m$$

and

$$G_n(u)=(1+u)\left(\sum_{k=0}^{m-1}X_ku^k\right)^2+(1-u)\left(\sum_{k=0}^{m-1}Y_ku^k\right)^2 \qquad \text{for} \qquad n=2m-1.$$

It thus follows from these representations that

$$\mathfrak{S}\{G_n\}=\sum_{0}^{m}s_{l+k}X_lX_k+\sum_{0}^{m-1}(s_{l+k}-s_{l+k+2})Y_lY_k$$

for $n = 2m$ and

$$\mathfrak{S}\{G_n\} = \sum_0^{m-1}(s_{i+k}+s_{i+k+1})\,X_iX_k + \sum_0^{m-1}(s_{i+k}-s_{i+k+1})\,Y_iY_k$$

for $n = 2m - 1$.

Remark. If we replace the interval $<-1, 1>$ with the interval $<a, b>$, then instead of the forms mentioned in the formulation of Theorem 19 we must introduce the forms

$$\sum_0^m s_{i+k}X_iX_k, \qquad -\sum_0^{m-1}\{abs_{i+k}-(a+b)s_{i+k+1}+s_{i+k+2}\}\,X_iX_k$$

for $n = 2m$ and the forms

$$\sum_0^{m-1}(s_{i+k+1}-as_{i+k})\,X_iX_k, \qquad \sum_0^{m-1}(bs_{i+k}-s_{i+k+1})\,X_iX_k$$

for $n = 2m - 1$.

Every representation of a real non-negative definite sequence $\{c_k\}$:

$$c_k = \sum_{j=1}^m \rho_j \cos kt_j \qquad (k = 0, 1, \cdots, n), \tag{8_1}$$

in which $\rho_j > 0$ and t_j are real $(j = 1, 2, \cdots, m)$ generates a representation of the corresponding sequence $\{s_k\}$:

$$s_k = \sum_{j=1}^m \rho_j \xi_j^k \qquad (k = 0, 1, \cdots, n), \tag{8_2}$$

in which the numbers ρ_j are the same as in (8_1) and

$$\xi_j = \cos t_j \qquad (j = 1, 2, \cdots, m). \tag{8_3}$$

Conversely, every representation (8_2) in which $\rho_j > 0$ and $-1 \le \xi_j \le 1$ $(j = 1, 2, \cdots, m)$ generates a representation (8_1) in which the same numbers ρ_j appear as in (8_2) and t_j $(j = 1, 2, \cdots, m)$ is defined from (8_3).

Actually it follows from (8_1) that for any quasi-polynomial

$$T(z) = \sum_{k=-n}^n A_k z^k \qquad (A_{-k} = A_k)$$

the equation

$$\mathfrak{S}\{T\} = \sum_{j=1}^m \rho_j T(e^{it_j})$$

holds.

Thus

$$s_k = \mathfrak{C}\left\{\frac{1}{2^k}\left(z+\frac{1}{z}\right)^k\right\} = \sum_{j=1}^{m} \rho_j \cos^k t_j \qquad (k = 0,\ 1, \ldots,\ n).$$

Let (8_2) be given; then for any quasi-polynomial $G(u)$ of degree $\leq n$:

$$\mathfrak{S}\{G\} = \sum_{j=1}^{m} \rho_j G(\cos t_j),$$

from which recalling $(3''')$ we find that

$$c_k = \mathfrak{C}\left\{\frac{1}{2}\left(z^k+\frac{1}{z^k}\right)\right\} = \mathfrak{C}\left\{H_k\left(\frac{z+z^{-1}}{2}\right)\right\} = \mathfrak{S}\{H_k(u)\} =$$
$$= \sum_{j=1}^{m} \rho_j H_k(\cos t_j) = \sum_{j=1}^{m} \rho_j \cos k t_j,$$

and the theorem is proved.

The remark we have made leads by Theorems 13, 14, 15, 16 to the following propositions.[1)]

Theorem 13a. *If a sequence*

$$s_0,\ s_1,\ \cdots,\ s_{2m-1}$$

is positive definite in the interval $< -1,\ 1>$, *then there exists a canonical representation*

$$s_k = \sum_{j=1}^{m} \rho_j \xi_j^k \qquad (k = 0,\ 1,\ \cdots,\ 2m-1),$$

where

$$\rho_j > 0 \qquad (j = 1,\ 2,\ \cdots,\ m)$$

and

$$-1 < \xi_1 < \xi_2 < \cdots < \xi_m < 1,$$

and also a canonical representation

$$s_k = \sum_{j=0}^{m} \rho_j' \xi_j'^k \qquad (k = 0,\ 1,\ \cdots,\ 2m-1),$$

where

$$\rho_j' > 0 \qquad (j = 0,\ 1,\ \cdots,\ m)$$

and

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1) Cf. A. Markov [13a] and K. Posse [15].

$$-1 = \xi_0' < \xi_1' < \cdots < \xi_{m-1}' < \xi_m' = 1;$$

and moreover

$$\xi_0' < \xi_1 < \xi_1' < \cdots < \xi_m < \xi_m'.$$

Theorem 14a–15a. *If a sequence*

$$s_0, s_1, \cdots, s_{2m-2}$$

is positive definite in the interval $< -1, 1>$ *then there exists an infinite set of canonical representations*

$$s_k = \sum_{j=1}^{m} \rho_j \xi_j^k \qquad (k = 0, 1, \cdots, 2m-2),$$

where

$$\rho_j > 0 \quad (j = 1, 2, \cdots, m)$$

and

$$-1 \leq \xi_1 < \xi_2 < \cdots < \xi_m \leq 1;$$

among these representations there exists one for which $\xi_1 = -1$, *and one for which* $\xi_m = 1$; *the systems* $\xi_1, \xi_2, \cdots, \xi_m$ *corresponding to two distinct representations alternate.*

Theorem 16a. *If a sequence*

$$s_0, s_1, \cdots, s_n$$

is non-negative in the interval $< -1, 1>$ *and has rank* $p < n+1$, *then there is uniquely determined a canonical representation*

$$s_k = \sum_{j=1}^{r} \rho_j \xi_j^k \qquad (k = 0, 1, \cdots, n), \tag{9}$$

where

$$\rho_j > 0 \qquad (j = 1, 2, \cdots, r)$$

and

$$-1 \leq \xi_1 < \xi_2 < \cdots < \xi_r \leq 1.$$

In this representation r *equals the rank of the form*

$$\sum_{0}^{m} s_{i+k} X_i X_k, \quad \textit{where} \quad m = \left[\frac{n}{2}\right],$$

and also equals the order of the last non-zero determinant of the sequence

$$D_0, D_1, \cdots, D_m$$

where (*cf.* §1)

$$D_q = \left| s_{i+k} \right|_0^q.$$

The last assertion needs some clarification. It follows from (9) that $r \leq m+1$ and

$$\sum_0^m s_{i+k} X_i X_k = \sum_{j=1}^r \rho_j (X_0 + X_1 \xi_j + \cdots + X_m \xi_j^m)^2, \tag{9'}$$

so that r equals the rank of the above form, and hence,[1)]

$$D_r = D_{r+1} = \cdots = D_m = 0.$$

On the other hand, setting $X_r = X_{r+1} = \cdots = X_m = 0$ in (9'), we find that the form

$$\sum_0^{r-1} s_{i+k} X_i X_k = \sum_{j=1}^r \rho_j (X_0 + X_1 \xi_j + \cdots + X_{r-1} \xi_j^{r-1})^2$$

is positive, and therefore

$$D_0 > 0, D_1 > 0, \cdots, D_{r-1} > 0.$$

3. It remains for us to give rules for calculation directly through $s_0, s_1, \cdots, s_n$ of the numbers ρ_j and ξ_j in Theorems 13a, 14a, 15a, in other words rules for the construction of quadrature formulas corresponding to the theorems we have demonstrated.

We shall give a summary of the quadrature formulas mentioned.

1) For any polynomial $G(u)$ of degree $\leq 2m-1$ (Theorem 13a):

$$\mathfrak{S}\{G(u)\} = \sum_{j=1}^m \rho_j G(\xi_j); \quad \rho_j > 0; \quad -1 < \xi_1 < \xi_2 < \cdots < \xi_m < 1; \tag{A}$$

$$\mathfrak{S}\{G(u)\} = \rho_0' G(-1) + \sum_{j=1}^{m-1} \rho_j' G(\xi_j') + \rho_m' G(1); \quad \rho_j' > 0; \tag{B}$$

$$-1 = \xi_0' < \xi_1' < \cdots < \xi_{m-1}' < \xi_m' = 1.$$

2) For any polynomial $H(u)$ of degree $\leq 2m-2$ (Theorems 14a–15a):

$$\mathfrak{S}\{H(u)\} = \rho_1 H(-1) + \sum_{j=2}^m \rho_j H(\xi_j); \ \rho_j > 0; \ -1 = \xi_1 < \xi_2 < \cdots < \xi_m < 1; \quad (C_{-1})$$

$$\mathfrak{S}\{H(u)\} = \sum_{j=1}^{m-1} \rho_j' H(\xi_j') + \rho_m' H(1); \ \rho_j' > 0; \ -1 < \xi_1' < \xi_2' < \cdots < \xi_m' = 1. \quad (C_{+1})$$

1) If $r = m+1$, this equation is eliminated.

We shall assume here that the sequence

$$s_0, s_1, \cdots, s_n, \tag{2}$$

where $n = 2m - 1$ (formulas (A), (B)) or $n = 2m - 2$ (formulas (C_{-1}), (C_{+1})) is positive definite in the interval $<-1, 1>$.

We shall begin with the quadrature formula (A).

The sequence (2), positive definite in the interval $<-1, 1>$, obviously is positive definite in every interval which includes the interval $<-1, 1>$ in particular on the interval $(-\infty, \infty)$. We may therefore apply Theorem 3 from the proof of which it follows that the numbers $\xi_1, \xi_2, \cdots, \xi_m$ are roots of the orthogonal polynomial

$$P_m(u) = \begin{vmatrix} s_0 & s_1 & \cdots & s_m \\ s_1 & s_2 & \cdots & s_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_{m-1} & s_m & \cdots & s_{2m-1} \\ 1 & u & \cdots & u^m \end{vmatrix}.$$

As far as the numbers ρ_j corresponding to them are concerned, they are easily calculated with the help of §1.

Let us turn to the quadrature formula (B).

Setting in this formula

$$G(u) = (1 - u^2) R_{m-1}(u) u^k \qquad (k = 0, 1, \cdots, m - 2),$$

where

$$R_{m-1}(u) = \text{const } (u - \xi_1')(u - \xi_2') \cdots (u - \xi_{m-1}'),$$

we find that

$$\mathfrak{S}\{(1 - u^2) R_{m-1}(u) u^k\} = 0 \qquad (k = 0, 1, \cdots, m - 2).$$

From this, introducing the operator $\mathfrak{S}^*$:

$$s_k^* = \mathfrak{S}^*\{u^k\} = \mathfrak{S}\{(1 - u^2) u^k\} = s_k - s_{k+2} \qquad (k = 0, 1, \cdots, 2m - 3),$$

we conclude that $R_{m-1}(u)$ is an orthogonal polynomial of degree $m - 1$ relative to the new sequence $s_0^*, s_1^*, \cdots, s_{2m-3}^*$, which obviously is also positive definite in the interval $<-1, 1>$.

Utilizing the result obtained in case (A), we find that the numbers $\xi_1', \xi_2', \cdots, \xi_{m-1}'$ are roots of the polynomial

$$R_{m-1}(u) = \begin{vmatrix} s_0 - s_2 & s_1 - s_3 & \cdots & s_{m-1} - s_{m+1} \\ s_1 - s_3 & s_2 - s_4 & \cdots & s_m - s_{m+2} \\ \cdots & \cdots & \cdots & \cdots \\ s_{m-2} - s_m & s_{m-1} - s_{m+1} & \cdots & s_{2m-3} - s_{2m-1} \\ 1 & u & \cdots & u^{m-1} \end{vmatrix}.$$

In order to find the numbers ρ_j', it is sufficient to set in formula (B)

$$G(u) = \frac{(1-u^2) R_{m-1}(u)}{u - \xi_i'} \qquad (i = 0, 1, 2, \ldots, m).$$

In this way we get

$$\rho_i' = \frac{1}{(1-\xi_i'^2) R_{m-1}'(\xi_i')} \mathfrak{S}\left\{ \frac{(1-u^2) R_{m-1}(u)}{u - \xi_i'} \right\} \qquad (i = 1, 2, \ldots, m-1)$$

$$\rho_0' = \frac{1}{2R_{m-1}(-1)} \mathfrak{S}\{(1-u) R_{m-1}(u)\};$$

$$\rho_m' = \frac{1}{2R_{m-1}(1)} \mathfrak{S}\{(1+u) R_{m-1}(u)\}.$$

Let us turn now to quadrature formulas (C_{-1}), (C_{+1}).

It is easy to see that it is now possible to apply Theorem 3 in accordance with which for any η, if $P_{m-1}(\eta) \neq 0$, there exists a unique canonical representation of a sequence which is positive definite in the interval $(-\infty, \infty)$ (as sequence (2) obviously is) and this system of numbers ξ_j contains the point η. We must only convince ourselves that $P_{m-1}(-1) \neq 0$, $P_{m-1}(1) \neq 0$ but this follows readily from the fact that sequence (2) is positive definite in the interval $<-1, 1>$. Indeed, if for example $P_{m-1}(1) = 0$, in view of the non-negativeness of the polynomial

$$\frac{P_{m-1}^2(u)}{1-u}$$

in the interval $<-1, 1>$ the inequality

$$\mathfrak{S}\left\{ \frac{P_{m-1}^2(u)}{1-u} \right\} > 0$$

must hold, which contradicts the condition of orthogonality.

Applying Theorem 3 and recalling again what was said at the end of §1 we find that $\xi_1, \xi_2, \cdots, \xi_m$ (formula (C_{-1})) are roots of the quasi-orthogonal polynomial

$$Q_m(u; -1) = Q_m(u)P_{m-1}(-1) - Q_m(-1)P_{m-1}(u);$$

in similar manner for the formula (C_{+1}) we find that the numbers ξ_j' are roots of the quasi-orthogonal polynomial

$$Q_m(u; 1) = Q_m(u)P_{m-1}(1) - Q_m(1)P_{m-1}(u).$$

Here $Q_m(u)$ is a real quasi-orthogonal polynomial of degree m.

4. If the sequence

$$s_0, s_1, \cdots, s_n \tag{2}$$

is positive definite in the interval $<-1, 1>$, then the extended sequence

$$s_0, s_1, \cdots, s_{n+1}$$

will be non-negative definite in the interval $<-1, 1>$ if and only if

$$s_{n+1}^{(g)} \leq s_{n+1} \leq s_{n+1}^{(d)}$$

where $s_{n+1}^{(g)}$ and $s_{n+1}^{(d)}$ depend only upon the numbers $s_0, s_1, \cdots, s_n$.

It is not difficult to find expressions for $s_{n+1}^{(g)}$ and $s_{n+1}^{(d)}$; namely for $n = 2m-1$

$$s_{2m}^{(g)} = -\frac{\begin{vmatrix} s_0 & \cdots & s_{m-1} & s_m \\ \cdots & \cdots & \cdots & \cdots \\ s_{m-1} & \cdots & s_{2m-2} & s_{2m-1} \\ s_m & \cdots & s_{2m-1} & 0 \end{vmatrix}}{\begin{vmatrix} s_0 & \cdots & s_{m-1} \\ \cdots & \cdots & \cdots \\ s_{m-1} & \cdots & s_{2m-2} \end{vmatrix}},$$

$$s_{2m}^{(d)} = \frac{\begin{vmatrix} s_0 - s_2 & \cdots & s_{m-2} - s_m & s_{m-1} - s_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_{m-2} - s_m & \cdots & s_{2m-4} - s_{2m-2} & s_{2m-3} - s_{2m-1} \\ s_{m-1} - s_{m+1} & \cdots & s_{2m-3} - s_{2m-1} & 0 \end{vmatrix}}{\begin{vmatrix} s_0 - s_2 & \cdots & s_{m-2} - s_m \\ \cdots & \cdots & \cdots \\ s_{m-2} - s_m & \cdots & s_{2m-4} - s_{2m-2} \end{vmatrix}} + s_{2m-2},$$

and for $n = 2m$

$$s^{(g)}_{2m-1} = -s_{2m-2} - \frac{\begin{vmatrix} s_0 + s_1 \cdots s_{m-1} + s_m \\ \cdots\cdots\cdots\cdots \\ s_{m-1} + s_m \cdots 0 \end{vmatrix}}{\begin{vmatrix} s_0 + s_1 \cdots\cdots s_{m-2} + s_{m-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ s_{m-2} + s_{m-1} \cdots s_{2m-4} + s_{2m-3} \end{vmatrix}},$$

$$s^{(d)}_{2m-1} = s_{2m-2} + \frac{\begin{vmatrix} s_0 - s_1 \cdots\cdots s_{m-1} - s_m \\ \cdots\cdots\cdots\cdots \\ s_{m-1} - s_m \cdots 0 \end{vmatrix}}{\begin{vmatrix} s_0 - s_1 \cdots\cdots s_{m-2} - s_{m-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ s_{m-2} - s_{m-1} \cdots s_{2m-4} - s_{2m-3} \end{vmatrix}}.$$

If we set $s_{n+1} = s^{(g)}_{n+1}$ or $s_{n+1} = s^{(d)}_{n+1}$, then the extended sequence will already not be positive definite in the interval $<-1, 1>$ but will have rank $n+1$.

Therefore, for the sequence extended in such a manner there will by Theorem 16 a be only one quadrature formula.

It is easy to see that we will obtain a quadrature formula of the type

(A), if $s_{2m} = s^{(g)}_{2m}$
(B), if $s_{2m} = s^{(d)}_{2m}$ $\Big\}$ $n = 2m-1$

(C_{-1}), if $s_{2m-1} = s^{(g)}_{2m-1}$
(C_{+1}), if $s_{2m-1} = s^{(d)}_{2m-1}$ $\Big\}$ $n = 2m-2.$

This consideration allows us to give a rule for finding the values ρ_j, ξ_j entering the canonical representation of a sequence $s_0, s_1, \cdots, s_n$ which is non-negative definite in the interval $<-1, 1>$ (cf. Theorem 16 a).

Having determined the rank $p < n+1$ of this sequence (with the help of the corollary to Theorem 18), we obtain the following possibilities:

1°, $p = 2m$, $\left|s_{i+k}\right|_0^m = 0$, i.e., $s_{2m} = s^{(g)}_{2m}$;

2°, $p = 2m$, $\left|s_{i+k} - s_{i+k+2}\right|_0^{m-1} = 0$, i.e., $s_{2m} = s^{(d)}_{2m}$;

3°, $p = 2m-1$, $\left|s_{i+k} + s_{i+k+1}\right|_0^{m-1} = 0$, i.e., $s_{2m-1} = s^{(g)}_{2m-1}$;

4°, $p = 2m-1$, $\left|s_{i+k} - s_{i+k+1}\right|_0^{m-1} = 0$, i.e., $s_{2m-1} = s^{(d)}_{2m-1}$.

To these cases there correspond quadrature formulas (canonical representations) of corresponding types (A), (B), (C_{-1}), (C_{+1}) and hence the determination of the quantities ρ_j, ξ_j is accomplished by the formulas displayed above.

CHAPTER II

EXPONENTIAL AND TRIGONOMETRIC MOMENT PROBLEMS

Let us concern ourselves with application of the theorems proved in the previous chapter to the solution of the classical moment problems and some problems of analysis closely connected with them.

We begin with trigonometric moment problems, first solved by F. Riesz [36 a] and Herglotz [30] in connection with a well-known problem of C. Carathéodory [22].

§1

1. From the results of Chapter I immediately follows

Theorem 1. *In order that there exist a non-decreasing function $\sigma(t)$, having exactly p points of growth and satisfying the conditions*

$$c_k = \int_0^{2\pi} e^{ikt} d\sigma(t) \qquad (k = 0, 1, 2, \cdots), \tag{1}$$

where the sequence $\{c_k\}$ has at least p elements, it is necessary and sufficient that the sequence $\{c_k\}$ be non-negative definite (on the circumference) and have rank p.

Proof. If the sequence $\{c_k\}$ is non-negative definite and has rank p, then on the basis of Theorem 12, Chapter I there is a uniquely determined representation

$$c_k = \sum_{j=1}^{p} \rho_j \alpha_j^k \qquad (k = 0, 1, 2, \cdots, p-1, \cdots),$$

where $\rho_j > 0$, $\alpha_j = e^{it_j}$, and moreover one may suppose that

$$0 \le t_1 < t_2 < \cdots < t_p < 2\pi.$$

Therefore, putting

$$\sigma(t) \equiv \sigma_p(t) = \sum_{t_j < t} \rho_j \qquad (0 \le t \le 2\pi),$$

we will find the desired representation (1).

Conversely, if representation (1) holds, then the sequence $\{c_k\}$ is obviously non-negative definite, since

$$\sum_{k=-m}^{m} A_k c_k = \int_0^{2\pi} \sum_{k=-m}^{m} A_k e^{ikt} d\sigma(t) \quad (A_{-k} = \bar{A}_k,\ c_{-k} = \bar{c}_k).$$

Since $\sigma(t)$ has exactly p points of growth, for $m \leq p-1$ it will always follow from

$$\sum_{k=-m}^{m} A_k e^{ikt} \geq 0 \qquad (0 \leq t \leq 2\pi,\ \Sigma |A_k| > 0) \tag{2}$$

that

$$\sum_{k=-m}^{m} A_k c_k > 0.$$

On the other hand, for $m \geq p$ the trigonometric polynomial (2) can always be chosen so that it equals zero at all points of growth of the function $\sigma(t)$ with the consequence that

$$\sum_{k=-m}^{m} A_k c_k = 0.$$

Theorem 2 (F. Riesz – G. Herglotz). *In order that there exist a non-decreasing function $\sigma(t)$ with an infinite set of points of growth, satisfying the condition*

$$c_k = \int_0^{2\pi} e^{ikt} d\sigma(t) \qquad (k = 0, 1, 2, \cdots),$$

it is necessary and sufficient that the sequence $\{c_k\}$ be positive definite.

Proof. The necessity of the condition is obtained in the same way as in the previous theorem.

For the proof of sufficiency we can suppose that the sequence $\{c_k\}$ is infinite. In fact, by Theorem 6, Chapter I, a positive definite sequence $c_0, c_1, \cdots, c_{n-1}$ can always be extended to a positive definite sequence $c_0, c_1, c_2, \cdots, c_n$.

According to Theorem 9, Chapter I, for any n there exists a representation

$$c_k = \sum_{j=1}^{n} \rho_{n,j} \alpha_{n,j}^k = \int_0^{2\pi} e^{ikt} d\sigma_n(t) \quad (k = 0, 1, \ldots, n-1),$$

where $\sigma_n(t)$ is defined as a certain step-function with n points of growth and moreover $\sigma_n(0) = 0$.

Since

$$0 \leq \sigma_n(t) \leq \int_0^{2\pi} d\sigma_n(t) = c_0 \qquad (0 \leq t \leq 2\pi),$$

Helly's theorem [1] can be applied to the sequence of functions $\{\sigma_n(t)\}$, in view of which there exist a subsequence $\{\sigma_{n_i}(t)\}$ and a non-decreasing function $\sigma(t)$, such that at every point of continuity of the function $\sigma(t)$:

$$\lim_{i \to \infty} \sigma_{n_i}(t) = \sigma(t).$$

By the second theorem of Helly [1], for every k:

$$\int_0^{2\pi} e^{ikt} d\sigma(t) = \lim_{j \to \infty} \int_0^{2\pi} e^{ikt} d\sigma_{n_j}(t) = c_k .$$

It remains to prove that $\sigma(t)$ has an infinite set of points of growth, but this follows from Theorem 1.

Theorem 3. *Let*

$$c_0, c_1, \cdots, c_{n-1}$$

be some positive definite sequence and

$$c_k = \sum_{j=1}^{n} \rho_j \alpha_j^k \qquad (k = 0, 1, \cdots, n-1;\ \rho_j > 0;\ \alpha_j = e^{it_j},\ 0 \le t_j < 2\pi) \tag{3}$$

any of its canonical representations. If the non-decreasing function $\sigma(t)$ satisfies the conditions

$$c_k = \int_0^{2\pi} e^{ikt} d\sigma(t) \qquad (k = 0, 1, \cdots, n-1) \tag{4}$$

and is different [2] *from $\sigma_n(t) = \sum_{t_j < t} \rho_j$, then in each of the intervals (t_j, t_{j+1}) $(j = 1, 2, \cdots, n;\ t_{n+1} = t_1 + 2\pi)$ there lies at least one point of growth of the function $\sigma(t)$, continued beyond the ends of the interval $\langle 0, 2\pi \rangle$ by the equality $\sigma(t + 2\pi) = c_0 + \sigma(t)$.*

Proof. The theorem to be proved is a generalization of Theorem 10, Chapter I, and can be obtained by the same methods.

We shall outline the reasoning for the interval (t_1, t_2), as an example.

Let

$$F(t) = \sin \frac{t - t_1}{2} \cdot \sin \frac{t - t_2}{2} \prod_{k=3}^{n} \sin^2 \frac{t - t_k}{2},$$

---

1) See for example V. Glivenko, *Stieltjes integral*, pp. 60–64.

2) We do not consider the functions $\sigma(t)$, $\sigma_n(t)$ distinct if they differ only by an additive constant at all of their points of continuity.

so that $F(t)$ is a trigonometric sum of order $n-1$, equal to zero at the points t_j. Therefore by representations (3) and (4)

$$\int_0^{2\pi} F(t)\,d\sigma(t) = \sum_{j=1}^{n} \rho_j F(t_j) = 0. \tag{5}$$

Note further that

$$F(t) \geq 0 \text{ for } t_2 < t < t_1 + 2\pi,$$

$$F(t) < 0 \text{ for } t_1 < t < t_2.$$

If the function $\sigma(t)$ did not have points of growth in the interval (t_1, t_2) then it would follow from (5) that

$$\int_0^{t_1} F(t)\,d\sigma(t) + \int_{t_2}^{2\pi} F(t)\,d\sigma(t) = 0;$$

but this equality is possible only if the only points of growth of the function $\sigma(t)$ are zeros of the function $F(t)$, which contradicts our condition.

Thus $\sigma(t)$ has at least one point of growth in the interval (t_1, t_2).

We shall agree to call the function $f(z)$ a Carathéodory function or a C-function for short if it is regular in the circle $|z| < 1$ and has a positive real part.

Concerning such functions the following theorem holds:

Theorem 4 (F. Riesz–Herglotz). *In order for a function $f(z)$ to be a C-function it is necessary and sufficient that it allow a representation*

$$f(z) = i\Im f(0) + \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\omega(t) \qquad (|z|<1),$$

where $\omega(t)$ is a non-decreasing function.

Proof. If

$$f(z) = c + c_1 z + c_2 z^2 + \cdots$$

is a C-function, then for any n and any $x_0, x_1, \cdots, x_n$:

$$\sum_{\alpha,\beta=0}^{n} c_{\alpha-\beta}\, x_\alpha \bar{x}_\beta = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \{f(re^{i\theta}) + \overline{f(re^{i\theta})}\}\,|x_0 + x_1 e^{i\theta} + \cdots + x_n e^{in\theta}|^2\,d\theta \geq 0 \qquad (c_0 = c + \bar{c})$$

and this means the sequence $\{c_k\}$ is non-negative definite. But by Theorems 1

and 2 there exists a non-decreasing function $\sigma(t)$ satisfying the equations

$$c_k = \int_0^{2\pi} e^{ikt} d\sigma(t) \qquad (k = 0, 1, 2, \cdots),$$

and thus

$$f(z) = c + c_1 z + c_2 z^2 + \cdots = \frac{c - \bar{c}}{2} + \frac{1}{2} \int_0^{2\pi} \frac{1 + e^{it} z}{1 - e^{it} z} d\sigma(t) \qquad (|z| < 1).$$

Conversely, if for some non-decreasing function $\sigma(t)$ the equation above holds, then $f(z)$ is obviously regular in the circle $|z| < 1$ and

$$\Re f(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta + t) + r^2} d\sigma(t) \geq 0.$$

The theorem is proved.

Comparing this theorem with Theorems 1 and 2 we find the following results:

Theorem 5a (Caratheodory–Toeplitz). *In order that a C-function $f(z)$ satisfying the condition*

$$f(z) = c + c_1 z + \cdots + c_{n-1} z^{n-1} + z^n \mathfrak{P}(z)$$

exist, it is necessary and sufficient that the sequence $\{c_k\}_0^{n-1}$ be non-negative definite ($c_0 = \bar{c} + c$). The C-function $f(z)$ is uniquely determined by its first coefficients $c_0, c_1, \cdots, c_{n-1}$ if and only if the rank p of the sequence $\{c_k\}_0^{n-1}$ is less than n, and in this and only this case it is a rational function of the form

$$\frac{c - \bar{c}}{2} + \frac{1}{2} \sum_{j=1}^{p} \frac{1 + e^{it_j z}}{1 - e^{it_j z}} \rho_j.$$

Let us clarify the last assertion of the theorem.

If the function $f(z)$ has the form

$$\frac{c - \bar{c}}{2} + \frac{1}{2} \sum_{j=1}^{p} \frac{1 + e^{it_j z}}{1 - e^{it_j z}} \rho_j,$$

then the coefficients c_k allow a representation

$$c_k = \sum_{j=1}^{p} \rho_j e^{ikt_j} \qquad (k = 0, 1, \cdots),$$

which means that the non-negative definite sequence has rank p.

Conversely, if the non-negative definite sequence $\{c_k\}$ has rank p, then by constructing the indicated canonical representation for it we find that

$$\frac{c-\bar{c}}{2}+\frac{1}{2}\sum_{j=1}^{p}\frac{1+e^{it_jz}}{1-e^{it_jz}}\rho_j = c+c_1z+\ldots+c_{n-1}z^{n-1}+z^n\mathfrak{P}(z).$$

The theorem is proved.

Theorem 5b (Carathéodory–Toeplitz). *For a power series*

$$c + c_1z + c_2z^2 + \cdots$$

to represent a C-function it is necessary and sufficient that the sequence $\{c_k\}$ $(c_0 = c + \bar{c})$ *be non-negative definite.*

2. The theorems we have obtained permit us to solve the exponential problem of moments for finite intervals, a problem studied by A. A. Markov [13a] long before Caratheodory and F. Riesz. (Actually, A. A. Markov never expressed his conditions in the language of forms or determinants.)

To this end it is enough to use Theorem 17 and the remark on page 33 about the solvability of relations (3) (page 31) for the quantities c_k. Thus, from the theorems we have proved in this section we have[1)]

Theorem 1a – 2a. *In order that there exist a non-decreasing function* $\sigma(u)$ *satisfying the equation*

$$s_k = \int_{-1}^{1} u^k\, d\sigma(u) \qquad (k = 0, 1, 2, \cdots)$$

it is necessary and sufficient that the sequence $\{s_k\}$ *be non-negative definite in the interval* $<-1, 1>$.

The function $\sigma(u)$ *has exactly* p *points of growth if and only if the non-negative definite sequence* $\{s_k\}$ *has rank* p.

3. If a non-negative definite sequence $\{c_k\}$ finite or infinite, on the circumference (or a non-negative definite sequence $\{s_k\}$ in the interval $<-1, 1>$) is not positive definite, the function $\sigma(t)$ of Theorem 1 (or the function $\sigma(u)$ of Theorem 1a–2a) is determined essentially uniquely.

On the other hand, if the sequence under examination is positive definite and finite, then the solution of the problem of moments is obviously not unique.

1) It is easy to see that Theorem 1a–2a may also be obtained by using the canonical representations of a sequence which is non-negative definite in the interval $<-1, 1>$, by reasoning like that outlined in the foregoing paragraph for the proof of Theorems 1 and 2.

The question of the uniqueness of a positive definite infinite sequence arises. The answer to this question is affirmative and follows from the following general proposition:

Theorem 6. *A function* $\sigma(t)$ $(0 \leq t \leq 2\pi)$ *of bounded variation* (*or* $\sigma(u)$, $-1 \leq u \leq 1$) *is determined in an essentially unique way by the moments*

$$c_k = \int_0^{2\pi} e^{ikt} d\sigma(t) \qquad (k = 0, 1, 2, \dots)$$

(*or the moments*

$$s_k = \int_{-1}^{1} u^k d\sigma(u) \qquad (k = 0, 1, 2, \dots)).$$

Proof. It is sufficient to prove the theorem for the case of trigonometric moments:

Expanding the function $\sigma(t)$ in a Fourier series which by Jordan's [1] theorem converges everywhere to

$$\frac{\sigma(t-0)+\sigma(t+0)}{2} \qquad (\sigma(-0)=\sigma(2\pi), \sigma(2\pi+0)=\sigma(0)),$$

we get

$$\frac{\sigma(t-0)+\sigma(t+0)}{2} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt).$$

Therefore by Abel's second theorem

$$\frac{\sigma(t-0)+\sigma(t+0)}{2} = \frac{a_0}{2} + \lim_{r \to 1-0} \sum_{k=1}^{\infty} r^k (a_k \cos kt + b_k \sin kt) =$$

$$= \lim_{r \to 1-0} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\,\sigma(\theta)\,d\theta}{1 - 2r\cos(t-\theta) + r^2}.$$

Let us consider the function

$$f(z) = \frac{c_0}{2} + c_1 z + c_2 z^2 + \dots = \frac{1}{2} \int_0^{2\pi} \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z}\, d\sigma(\theta) \qquad (|z| < 1).$$

For this function

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1) See for example de la Vallée–Poussin, *Cours d'analyse*, volume II, page 135.

$$\int_\theta^t \Re\{f(re^{-i\tau})\}\,d\tau = \frac{1}{2}\int_0^{2\pi} d\sigma(\theta)\int_0^t \frac{(1-r^2)\,d\tau}{1-2r\cos(\theta-\tau)+r^2} =$$

$$= \text{const} + \frac{1}{2}\int_0^{2\pi} \frac{(1-r^2)\,\sigma(\theta)\,d\theta}{1-2r\cos(t-\theta)+r^2}.$$

Comparison with the formula found earlier gives

$$\frac{\sigma(t-0)+\sigma(t+0)}{2} = \text{const} + \lim_{r\to 1-0}\frac{1}{\pi}\int_\theta^t \Re\{f(re^{-i\tau})\}\,d\tau.$$

From this inversion formula our assertion follows.

Remark. The relations we have established allow us to assert that a function of bounded variation has a constant value on some open arc $\alpha < t < \beta$ if and only if the corresponding function

$$f(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+e^{i\theta}z}{1-e^{i\theta}z}\,d\sigma(\theta)$$

is regular on this arc and has purely imaginary values. The necessity follows from the integral representation and the sufficiency from the inversion formula.

§2

1. In this paragraph we introduce several results relating to the exponential problem of moments on an infinite interval. For the case of the interval $(0, \infty)$ this problem was first raised and studied in complete detail by T. Stieltjes [39a]; for the case of the interval $(-\infty, \infty)$ it was studied comparatively recently in papers by Hamburger [29], M. Riesz [37], T. Carleman [29], and R. Nevanlinna [34a].

The central question in the researches of these authors is, "When is the problem of moments determined, i.e., when is the solution essentially unique?" Since this question is incidental to our investigations, we cannot consider it here but shall restrict ourselves in this paragraph to a presentation of the simplest theorems, analogous to the theorems of §1.

Moreover, in order not to complicate the arguments with the consideration of cases of degeneracy which were actually studied in Chapter I, we shall require $\sigma(u)$ to have an infinite number of growth points.

Theorem 7. *In order that there exist a non-decreasing function* $\sigma(u)$ *with an*

infinite set of growth points satisfying the equations

$$s_k = \int_{-\infty}^{\infty} u^k d\sigma(u) \qquad (k = 0, 1, 2, \cdots)$$

it is necessary and sufficient that the sequence $\{s_k\}$ *(finite or infinite) be positive definite in the interval* $(-\infty, \infty)$.

The proof is analogous to that of Theorem 2.

We shall spend time only on the passage to the limit.

Applying Theorem 3 of Chapter I, we find for every n some non-decreasing function $\sigma_n(u)$ with n growth points for which

$$s_k = \int_{-\infty}^{\infty} u^k d\sigma_n(u) \qquad (k = 0, 1, \cdots, 2n-2;\ n = 1, 2, 3, \cdots;\ \sigma_n(-\infty) = 0),$$

such that

$$0 \leq \sigma_n(u) \leq s_0 \qquad (n = 1, 2, \cdots).$$

According to Helly's theorem there exists a non-decreasing function $\sigma(u)$ and a sequence $\{n_i\}$ for which

$$\lim_{i \to \infty} \sigma_{n_i}(u) = \sigma(u)$$

at all points of continuity of the function $\sigma(u)$.

It remains to show that for any k:

$$s_k = \int_{-\infty}^{\infty} u^k d\sigma(u).$$

Let $2r - 2 > k$ so that for $n_i > r$:

$$s_k = \int_{-\infty}^{\infty} u^k d\sigma_{n_i}(u), \qquad s_{2r-2} = \int_{-\infty}^{\infty} u^{2r-2} d\sigma_{n_i}(u).$$

Further let $-A$, B $(A > 0,\ B > 0)$ be points of continuity of $\sigma(u)$. Then

$$\left| s_k - \int_{-A}^{B} u^k d\sigma_{n_i}(u) \right| \leqslant \int_{-\infty}^{-A} |u|^k d\sigma_{n_i}(u) + \int_{B}^{\infty} u^k d\sigma_{n_i}(u) \leqslant$$

$$\leqslant \frac{1}{M^{2r-2-k}} \left\{ \int_{-\infty}^{-A} u^{2r-2} d\sigma_{n_i}(u) + \int_{B}^{\infty} u^{2r-2} d\sigma_{n_i}(u) \right\} \leqslant \frac{s_{2r-2}}{M^{2r-2-k}},$$

where $M = \min\{A, B\}$.

Therefore, applying the second theorem of Helly we obtain

$$\left| s_k - \int_{-A}^{B} u^k d\sigma(u) \right| \leq \frac{s_{2r-2}}{M^{2r-2-k}},$$

and then letting A and B tend to infinity we find

$$s_k = \int_{-\infty}^{\infty} u^k d\sigma(u).$$

2. Let us agree to call a function $F(z)$ a Nevanlinna function or an N-function for short if it is regular in the half plane $\Im z > 0$ and has a negative imaginary part in that region.

Theorem 8. *Every N-function $F(z)$ admits a representation*

$$F(z) = \alpha - \mu z + \int_{-\infty}^{\infty} \frac{1 + zt}{z - t} d\sigma(t),$$

where $\sigma(t)$ is a non-decreasing function and $\mu \geq 0$ and α are two real constants.

Proof. Setting

$$z = i \frac{1 + \zeta}{1 - \zeta}, \quad f(\zeta) = iF(z),$$

we find that $f(\zeta)$ is a C-function.

Therefore according to Theorem 5 there exists a non-decreasing function $\phi(\tau)$ such that

$$f(\zeta) = i\Im f(0) + \int_0^{2\pi} \frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta} d\phi(\tau) \qquad (|\zeta| < 1)$$

or

$$f(\zeta) = i\Im f(0) + \mu \frac{1 + \zeta}{1 - \zeta} + \int_{+0}^{2\pi - 0} \frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta} d\phi(\tau),$$

where

$$\mu = \{\phi(+0) - \phi(0)\} + \{\phi(2\pi) - \phi(2\pi - 0)\} \geq 0.$$

Turning from ζ to z we find

$$iF(z) = i\Im\{iF(i)\} - i\mu z + i \int_{+0}^{2\pi - 0} \frac{1 - z \operatorname{ctg} \frac{\tau}{2}}{z + \operatorname{ctg} \frac{\tau}{2}} d\phi(\tau),$$

after which it remains to set

$$\Im\{iF(i)\} = \alpha$$

and perform the substitution

$$\operatorname{ctg}\frac{\tau}{2} = -t, \quad d\phi(\tau) = d\sigma(t).$$

Remark 1. It is not hard to see that the function $\sigma(t)$ appearing in the formulation of Theorem 8 is determined essentially uniquely for a given N-function $F(z)$.[1)]

It follows by Theorem 6 that the function $\phi(\tau)$ and therefore $\sigma(t)$ $(\operatorname{ctg}\frac{\tau}{2} = -t)$ is essentially uniquely defined by the function

$$f(\zeta) = iF\left(i\,\frac{1+\zeta}{1-\zeta}\right).$$

We also note that by the note to Theorem 6 the function $\sigma(t)$ of Theorem 8 is constant in the open interval $\alpha < t < \beta$ if and only if $F(z)$ is regular in this interval and real.

Remark 2. Using Theorem 8 it is not hard to prove that for every N-function $F(z)$ and some sufficiently small but fixed $\delta > 0$

$$\lim_{|z|\to\infty} \frac{F(z)}{z} \qquad (\delta \le \arg z \le \pi - \delta)$$

exists and is the number μ in the integral representation.

Theorem 9. *If an N-function $F(z)$ satisfies the equation*

$$\lim_{y\to+\infty} (iy)^{2n+1}\left\{F(iy) - \frac{s_0}{iy} - \frac{s_1}{(iy)^2} - \cdots - \frac{s_{2n}}{(iy)^{2n+1}}\right\} = 0, \tag{1}$$

where the numbers $s_1, s_3, s_5, \cdots, s_{2n-1}$ are real, then $F(z)$ admits a representation

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1) We may also show immediately that the following inversion formula holds:

$$\frac{\psi(\beta+0)+\psi(\beta-0)}{2} - \frac{\psi(\alpha+0)+\psi(\alpha-0)}{2} =$$

$$= \lim_{\eta\to 0}\frac{1}{\pi}\int_{\alpha+i\eta}^{\beta+i\eta} \Im\{F(t+i\eta)\}\,dt,$$

where

$$\psi(t) = \int_c^t (1+u^2)\,d\sigma(u).$$

In a certain more special case this inversion formula was first proved by Stieltjes [39a].

$$F(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{z - t},$$

where $\omega(t)$ *is a non-decreasing function satisfying the equations*

$$s_k = \int_{-\infty}^{\infty} t^k d\omega(t) \qquad (k = 0, 1, \cdots, 2n),$$

such that all the numbers s_k *are real and the sequence* $s_0, s_1, \cdots, s_{2n}$ *is non-negative definite in the interval* $(-\infty, \infty)$.

Proof. On the basis of Theorem 8, $F(z)$ admits a representation

$$F(z) = \alpha - \mu z + \int_{-\infty}^{\infty} \frac{1 + tz}{z - t}\, d\sigma(t) \qquad (\mu \geq 0)$$

which means that

$$\Re\{iyF(iy)\} = \mu y^2 + \int_{-\infty}^{\infty} \frac{y^2(1+t^2)}{t^2 + y^2}\, d\sigma(t), \tag{2}$$

$$\Im\{iyF(iy)\} = \alpha y - \int_{-\infty}^{\infty} \frac{ty(1-y^2)}{t^2 + y^2}\, d\sigma(t). \tag{3}$$

By (1):

$$\lim_{y \to +\infty} iyF(iy) = s_0; \tag{4}$$

therefore both expressions (2) and (3) have limits for $y \to \infty$ which means that

$$\mu = 0, \quad \alpha = \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{t(1-y^2)}{t^2 + y^2}\, d\sigma(t),$$

$$\lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{(1+t^2)y^2}{t^2 + y^2}\, d\sigma(t) = s_0'.$$

The previous equality shows that for any $A > 0$, $B > 0$:

$$\int_{-A}^{B} (1 + t^2)\, d\sigma(t) = \lim_{y \to \infty} \int_{-A}^{B} \frac{(1 + t^2)\, d\sigma(t)}{1 + \frac{t^2}{y^2}} \leqslant s_0'.$$

Therefore the integral

$$\int_{-\infty}^{\infty} (1+t^2)\, d\sigma(t)$$

exists and this means that the integral

$$\int_{-\infty}^{\infty} |t| d\sigma(t)$$

also exists.

Thus

$$\alpha = \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{t(1-y^2)}{t^2+y^2} d\sigma(t) = -\int_{-\infty}^{\infty} t d\sigma(t)$$

and setting

$$\omega(t) = \int_{-\infty}^{t} (1+t^2) d\sigma(t),$$

we find that

$$F(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{z-t}.$$

Hence

$$\left| iy F(iy) - \int_{-A}^{B} \frac{d\omega(t)}{1-\frac{t}{iy}} \right| \leqslant \int_{B}^{\infty} d\omega(t) + \int_{-\infty}^{-A} d\omega(t) = \varepsilon_{A,B}$$

and by (4):

$$\left| s_0 - \int_{-A}^{B} d\omega(t) \right| \leqslant \varepsilon_{A,B},$$

so that

$$s_0 = \int_{-\infty}^{\infty} d\omega(t).$$

We now assume that the equalities

$$s_k = \int_{-\infty}^{\infty} t^k d\omega(t)$$

are proved for $k = 0, 1, 2, \cdots, 2m$, where $m \leq n-1$ and shall prove that they are also valid for $k = 2m+1,\ 2m+2$.

Since these equalities are valid for $m = 0$, in this way the proof of the theorem will be complete.

By virtue of our assumption:

$$F(z) = \sum_{k=0}^{2m} \frac{s_k}{z^{k+1}} + \frac{1}{z^{2m+1}} \int_{-\infty}^{\infty} \frac{t^{2m+1} d\omega(t)}{z-t}.$$

Therefore on the basis of (1):

$$\lim_{y \to +\infty} (iy)^2 \left\{ \int_{-\infty}^{\infty} \frac{t^{2m+1} d\omega(t)}{iy - t} - \frac{s_{2m+1}}{iy} - \frac{s_{2m+2}}{(iy)^2} \right\} = 0$$

or

$$\int_{-\infty}^{\infty} \frac{t^{2m+1} d\omega(t)}{iy - t} = \frac{s_{2m+1}}{iy} + \frac{s_{2m+2}}{(iy)^2} + \frac{\delta(y)}{y^2}, \tag{5}$$

where $\lim_{y \to +\infty} \delta(y) = 0$.

Taking real parts in the equation (5), and recalling that s_{2m+1} is real, gives

$$\int_{-\infty}^{\infty} \frac{t^{2m+2} d\omega(t)}{t^2 + y^2} = \frac{s'_{2m+2}}{y^2} + \frac{\delta_1(y)}{y^2},$$

where $\lim_{y \to +\infty} \delta_1(y) = 0$ and $s'_{2m+2} = \Re s_{2m+2}$.

From this equation we obtain by reasoning analogous to the above that the integral

$$\int_{-\infty}^{\infty} t^{2m+2} d\omega(t)$$

exists. Therefore the integral

$$\int_{-\infty}^{\infty} |t|^{2m+1} d\omega(t)$$

also exists and by (5):

$$s_{2m+1} = \int_{-\infty}^{\infty} t^{2m+1} d\omega(t),$$

$$s_{2m+2} = \int_{-\infty}^{\infty} t^{2m+2} d\omega(t).$$

Theorem 10. *If $\omega(t)$ is a non-decreasing function for which*

$$\int_{-\infty}^{\infty} t^k d\omega(t) = s_k \qquad (k = 0, 1, \cdots, 2n),$$

then

$$F(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{z - t} \tag{6}$$

is an N-function which for some $\delta > 0$, fixed but as small as desired, satisfies the equation

$$\lim_{|z| \to \infty} z^{2n+1} \left\{ F(z) - \frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n}}{z^{2n+1}} \right\} = 0 \qquad (\delta \leq \arg z \leq \pi - \delta). \tag{7}$$

Proof. That function (6) is an N-function is clear. Let us prove (7). For this we must obviously show that

$$\lim_{|z|\to\infty} \int_{-\infty}^{\infty} \frac{t^{2n+1}}{z-t}\, d\omega(t) = 0 \qquad (\delta \le \arg z \le \pi - \delta).$$

Noting that for $\delta \le \arg z \le \pi - \delta$ the inequality

$$\left|\frac{t}{z-t}\right| \le \frac{1}{\sin\delta}, \quad \left|\frac{z}{z-t}\right| \le \frac{1}{\sin\delta}$$

holds, we may write

$$\left|\int_{-\infty}^{\infty} \frac{t^{2n+1}\, d\omega(t)}{z-t}\right| \le \int_{-A}^{B} \frac{|t|^{2n+1}\, d\omega(t)}{|z|\sin\delta} + \frac{1}{\sin\delta}\int_{-\infty}^{-A} t^{2n} d\omega(t) +$$

$$+ \frac{1}{\sin\delta}\int_{B}^{\infty} t^{2n}\, d\omega(t),$$

from which

$$\overline{\lim_{|z|\to\infty}} \left|\int_{-\infty}^{\infty} \frac{t^{2n+1}\, d\omega(t)}{z-t}\right| \le \frac{1}{\sin\delta}\left\{\int_{-\infty}^{-A} t^{2n}\, d\omega(t) + \int_{B}^{\infty} t^{2n}\, d\omega(t)\right\} =$$

$$= \frac{1}{\sin\delta}\left\{s_{2n} - \int_{-A}^{B} t^{2n}\, d\omega(t)\right\}.$$

The right part tends to zero for $A, B \to \infty$, which proves the theorem.

Remark. From Theorem 10 and the proof of Theorem 9 it follows that an N-function $F(z)$ admits a representation

$$F(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{z-t},$$

where $\omega(t)$ is a non-decreasing bounded function if and only if

$$\overline{\lim_{y\to+\infty}} |yF(iy)| < \infty.$$

3. We shall supplement Theorems 9 and 10 with the following proposition:

Theorem 11. *In order that function $F(z)$ admit the representation*

$$F(z) = \int_{a}^{b} \frac{d\omega(t)}{z-t} + \text{const}$$

with $\omega(t)$ a non-decreasing function it is necessary and sufficient that the following conditions are fulfilled:

1) *$F(z)$ is regular outside the segment $a \leq x \leq b$,*

2) *$F(z) - F(\infty)$ is real for $z = x < a$ and $z = x > b$,*

3) *$F(z) - F(\infty)$ is an N-function.*

Proof. The necessity of the conditions is immediately verified. For the proof that these conditions are sufficient we may without loss of generality take

$$a = -1, \quad b = 1.$$

By assumption

$$F(z) - F(\infty) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots,$$

where the numbers $s_0, s_1, s_2, \cdots$ are real and

$$\overline{\lim_{n \to \infty}} \sqrt[n]{|s_n|} \leq 1. \tag{8}$$

Furthermore, according to Theorem 9 there exists a non-decreasing function $\omega(t)$ for which

$$F(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{z - t} + \text{const},$$

$$s_k = \int_{-\infty}^{\infty} t^k d\omega(t) \qquad (k = 0, 1, 2, \cdots),$$

and we must now prove that $\omega(t)$ is a constant in the open intervals [1] $(-\infty, -1)$, $(1, \infty)$.

Assuming the contrary, we suppose that $\omega(t)$ has some growth point t_0 lying outside the interval $< -1, 1 >$, for example $t_0 = 1 + 2\epsilon$, where $\epsilon > 0$.

Then

$$s_{2k} = \int_{-\infty}^{\infty} t^{2k} d\omega(t) \geq \int_{1+\epsilon}^{1+3\epsilon} t^{2k} d\omega(t) \geq (1+\epsilon)^{2k} \int_{1+\epsilon}^{1+3\epsilon} d\omega(t) = \delta(1+\epsilon)^{2k}$$
$$(k = 0, 1, 2, \cdots),$$

where $\delta > 0$.

---

1) This condition follows at once from the regularity and realness of the function $F(z) - F(\infty)$ for $z = x < -1$ and $z = x > 1$, in agreement with the note to Theorem 8 which was obtained from the inversion formula. The following reasoning does not depend upon the inversion formula.

Thus

$$\overline{\lim_{k\to\infty}}\sqrt[2k]{s_{2k}} \geq 1 + \epsilon,$$

which is impossible by (8).

CHAPTER III

THE L-MOMENT PROBLEM AND SOME OF ITS APPLICATIONS

In the present chapter we shall consider moment problems (trigonometric and exponential) with the supplementary requirement that there exists a bounded derivative for the desired non-decreasing function $\omega(t)$:

$$\begin{aligned} &d\omega(t) = f(t)\,dt, \\ &0 \leq f(t) \leq L. \end{aligned} \tag{1}$$

For solving the problem stated, the specific nature of inequality (1) often is inessential and we shall also consider conditions of a more general form:

$$\alpha \leq f(t) \leq \beta,$$

where α, β are given numbers. We then have, as a special case the condition

$$-L \leq f(t) \leq L.$$

Academician A. A. Markov [13b–d] first studied similar problems.

We shall call these problems L-moment problems.

§1

1. We shall begin with the trigonometric L-problem.

Theorem 1. *In order that there exist a pieceuise continuous function $f(t)$ satisfying the relations* [1)]

$$\begin{aligned} &-L \leq f(t) \leq L \qquad (0 \leq t \leq 2\pi), \\ &c_k = \int_0^{2\pi} f(t)\, e^{ikt}dt \qquad (k = 0, 1, 2, \cdots, n-1), \end{aligned} \tag{2}$$

it is necessary and sufficient that the inequality

$$-2\pi L \leq c_0 \leq 2\pi L \tag{3}$$

1) We shall consider here only trigonometric integrals from 0 to 2π. However, with the aid of non-trivial reasoning [1b] we may obtain a generalization of all our trigonometric problems to the case when the integral is taken from 0 to τ $(0 < \tau < 2\pi)$.

is satisfied and that the sequence

$$\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_{n-1},$$

defined by the expansion

$$\exp\{\frac{i}{2L}(\frac{c_0}{2} + c_1 z + c_2 z^2 + \cdots + c_{n-1} z^{n-1})\} = \gamma + \gamma_1 z + \gamma_2 z^2 + \cdots + \gamma_{n-1} z^{n-1} + \cdots,$$

$$\gamma_0 = \gamma + \bar{\gamma} = 2\cos\frac{c_0}{4L} \tag{4}$$

is non-negative definite on the circumference.

If these conditions are fulfilled, there exists a piecewise constant function $f_L(t)$ *satisfying relations* (2) *with period* 2π, *assuming both of the values* L *and* $-L$ *and having not more than* $2n$ *points of discontinuity in an interval of length* 2π.

Proof. 1. If a function $f(t)$ exists which satisfies relations (2), then the function

$$\Phi(z) = \frac{1}{4L}\int_0^{2\pi} \frac{1+e^{it}z}{1-e^{it}z} f(t)\,dt = \frac{1}{2L}(\frac{c_0}{2} + c_1 z + c_2 z^2 + \cdots + c_{n-1} z^{n-1} + \cdots)$$

is regular in the circle $|z| < 1$ and its real part

$$\Re\Phi(z) = \frac{1}{4L}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t+\theta)+r^2} f(t)\,dt \qquad (z = re^{i\theta})$$

does not exceed $\frac{1}{2}\pi$ in absolute value in this circle.

Therefore, setting

$$F(z) = e^{i\Phi(z)},$$

we find that for $|z| < 1$:

$$\Re F(z) = |F(z)|\cos\{\Re\Phi(z)\} \geq 0,$$

i.e., $F(z)$ is a C-function.

But since

$$F(z) = \gamma + \gamma_1 z + \gamma_2 z^2 + \cdots + \gamma_{n-1} z^{n-1} + \cdots,$$

by Theorem 4 of Chapter II the sequence

$$\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_{n-1} \qquad (\gamma_0 = \gamma + \bar{\gamma})$$

is non-negative definite on the circle.

The necessity of condition (3) is obvious.

2. We turn to the proof of the sufficiency of the conditions.

We shall assume that

$$|c_0| < 2\pi L, \tag{3'}$$

since for $c_0 = \pm 2\pi L$ the equality $\gamma_0 = 0$ holds, and by the non-negativeness of the sequence $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ this means that $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ are also equal to zero, which implies the equations $c_1 = c_2 = \cdots = c_{n-1} = 0$, and we obtain the trivial solution $f(t) \equiv \pm L$.

If the sequence

$$\gamma_0 > 0, \ \gamma_1, \gamma_2, \cdots, \gamma_{n-1}$$

is non-negative definite on the circumference, then by Theorems 9 and 12 of Chapter I there exists a representation

$$\gamma_k = \rho_1 \alpha_1^k + \rho_2 \alpha_2^k + \cdots + \rho_p \alpha_p^k \qquad (k = 0, 1, 2, \cdots, n-1),$$

where $p \leq n$, $\rho_j > 0$, $\alpha_j = e^{it_j}$, and we may assume that

$$t_1 < t_2 < \cdots < t_p < t_1 + 2\pi.$$

We shall construct the function

$$F(z) = \frac{\gamma - \bar{\gamma}}{2} + \frac{1}{2} \sum_{j=1}^{p} \rho_j \frac{1 + \alpha_j z}{1 - \alpha_j z}.$$

It is easy to see that

$$F(z) = \gamma + \gamma_1 z + \gamma_2 z^2 + \cdots + \gamma_{n-1} z^{n-1} + \cdots$$

is a rational function whose numerator and denominator are polynomials of degree p.

But since

$$F_1(t) = iF(e^{it}) = i\frac{\gamma - \bar{\gamma}}{2} - \frac{1}{2} \sum_{j=1}^{p} \rho_j \operatorname{ctg} \frac{t + t_j}{2},$$

we have that for small enough $\epsilon > 0$ (the numbers ρ_j being positive):

$$F_1(-t_k + \epsilon) < 0, \ F_1(-t_k - \epsilon) > 0 \qquad (k = 1, 2, \cdots, p).$$

Thus the zeros of the function $F(z)$ lie on the circle $|z| = 1$ and alternate with the points $\bar{\alpha}_j$.

Therefore

$$F(z) = A \prod_{k=1}^{p} \frac{e^{-\frac{i}{2}\tau_k} - e^{\frac{i}{2}\tau_k} z}{e^{-\frac{i}{2}t_k} - e^{\frac{i}{2}t_k} z},$$

where we may assume that

$$\tau_1 < t_1 < \tau_2 < \cdots < t_p < \tau_1 + 2\pi.$$

But since $|F(0)| = |\gamma| = 1$, we have $|A| = 1$; and since the function $F(z)$ has purely imaginary values on the circumference $|z| = 1$, we also have $A = \pm i$; on the other hand, since

$$\Re F(0) = \gamma_0 > 0$$

and

$$F(0) = \pm ie^{\frac{i}{2}\sum_{k=1}^{p}(t_k - \tau_k)},$$

besides which

$$0 < \frac{1}{2}\sum_{k=1}^{p}(t_k - \tau_k) < \pi,$$

we therefore have $A = -i$, which means

$$F(z) = -i\prod_{k=1}^{p}\frac{e^{-\frac{i}{2}\tau_k} - e^{\frac{i}{2}\tau_k}z}{e^{-\frac{i}{2}t_k} - e^{\frac{i}{2}t_k}z}.$$

Consider the function

$$f_L(t) = \begin{cases} L & (\tau_j < t < t_j) \\ -L & (t_j < t < \tau_{j+1}) \end{cases} \qquad (j = 1, 2, \cdots, p;\ \tau_{p+1} = \tau_1 + 2\pi)$$

$$f_L(t + 2\pi) = f_L(t).$$

This function may also be represented in the form

$$f_L(t) = -L\operatorname{sign}\prod_{k=1}^{p}\frac{\sin\frac{t-\tau_k}{2}}{\sin\frac{t-t_k}{2}}.$$

Direct calculation show that

$$\frac{1}{4L}\int_0^{2\pi}\frac{1+e^{it}z}{1-e^{it}z}f_L(t)\,dt = \frac{1}{i}\log F(z), \tag{5}$$

where that branch of the logarithm is chosen for which

$$\frac{1}{i}\log F(0) = \frac{1}{4L}\int_0^{2\pi} f_L(t)\,dt = \sum_{k=1}^{p}\frac{t_k - \tau_k}{2} - \frac{\pi}{2},$$

which means that it equals $\frac{c_0}{4L}$ by (3′). But since

$$F(z) = \gamma + \gamma_1 z + \cdots + \gamma_{n-1} z^{n-1} + \cdots = e^{\frac{i}{2L}} \left(\frac{c_0}{2} + c_1 z + \cdots + c_{n-1} z^{n-1} + \cdots \right),$$

we have by (5)

$$\frac{1}{2} \int_0^{2\pi} \frac{1 + e^{it} z}{1 - e^{it} z} f_L(t)\, dt = \frac{c_0}{2} + c_1 z + \cdots + c_{n-1} z^{n-1} + \cdots,$$

from which it follows that

$$c_k = \int_0^{2\pi} f_L(t) e^{ikt} dt \qquad (k = 0, 1, 2, \cdots, n-1).$$

Thus, the piecewise constant function

$$f_L(t) = -L \operatorname{sign} \prod_{k=1}^{p} \frac{\sin \frac{t - \tau_k}{2}}{\sin \frac{t - t_k}{2}} = L \operatorname{sign} \left\{ \frac{1}{i} F(e^{-it}) \right\},$$

where

$$F(z) = \frac{\gamma - \bar{\gamma}}{2} + \frac{1}{2} \sum_{j=1}^{p} \rho_j \frac{1 + \alpha_j z}{1 - \alpha_j z} = \gamma + \gamma_1 z + \cdots + \gamma_{n-1} z^{n-1} + \cdots$$

satisfies the requirements. This completes the proof of the theorem.

Remark. Later we shall often deal with functions defined in some interval which like the function $f_L(t)$ take on only the values L and $-L$ (or the two values $-L(1-\theta)$ and $L(1+\theta)$) and which have a specified number q of points of discontinuity; for conciseness we shall call such functions L-functions (or L_θ-functions) of order q.

We note furthermore that the L-function $f_L(t)$ constructed for the proof of Theorem 1 is the unique solution of the L-problem (2) if and only if the rank p of the sequence

$$\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$$

is less than n.

Indeed, from the above proof and from Theorem 9, Chapter I, it follows that for $p = n$ there exists an infinity of solutions, appearing as L-functions of order $2n$.

Conversely, because two distinct [1] solutions of the L-problem (2) allow us to extend the sequence

---

1) We shall not consider distinct functions which coincide almost everywhere.

$$\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$$

to an infinite non-negative definite sequence in two distinct ways, for $p < n$ the L-problem (2) cannot have two distinct solutions.

Moreover, it is not difficult to prove directly that if L-problem (2) has as a solution an L-function of order $2p < 2n$, then it is the unique solution to problem (2).

Indeed, let $f_L(t)$ have the points of discontinuity

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{2p} \qquad (0 < \alpha_1, \alpha_{2p} < 2\pi) \tag{6}$$

and let $f(t)$ be a solution of problem (2).

Then

$$\int_0^{2\pi} \{F_L(t) - f(t)\} T(t)\, dt = 0 \tag{7}$$

for any trigonometric sum $T(t)$ of order $\leq n-1$.

Taking

$$T(t) = \prod_{k=1}^{2p} \sin \frac{t-\alpha_k}{2} \qquad (p < n)$$

and noting that $f_L(t) - f(t)$ changes sign at the points (6), i.e., where $T(t)$ does, we obtain from (7) that

$$f_L(t) - f(t) = 0$$

at almost all points where the function $f_L(t)$ is continuous, which means that $f_L(t)$ is the unique solution of problem (2).

Theorem 2. *In order that there exist a measurable function $f(t)$ satisfying the relations*

$$\begin{gathered} -L \leq f(t) \leq L, \\ c_k = \int_0^{2\pi} f(t) e^{ikt} dt \qquad (k = 0, 1, 2, \cdots), \end{gathered} \tag{2}$$

it is necessary and sufficient that

$$-2\pi L \leq c_0 \leq 2\pi L$$

and that the sequence

$$\gamma_0, \gamma_1, \gamma_2, \cdots$$

be non-negative definite on the circumference, being defined with the help of the expansion

$$\exp\left\{\frac{i}{2L}\left(\frac{c_0}{2}+c_1z+\ldots+c_{n-1}z^{n-1}+\ldots\right)\right\}=\gamma+\gamma_1z+\cdots+\gamma_{n-1}z^{n-1}+\cdots$$

$$\gamma_0=\gamma+\bar{\gamma}=2\cos\frac{c_0}{4L}.$$

Proof. The necessity is proved in the same manner as in the previous theorem. The sufficiency may be verified with the aid of a simple limit process.

In fact, by Theorem 1, for any n there exists a function $f_L(t;\, n)$ satisfying the conditions

$$-L\leq f_L(t;\, n)\leq L,$$

$$c_k=\int_0^{2\pi} f_L(t;\, n)e^{ikt}dt \qquad (k=0,\, 1,\, \cdots,\, n-1).$$

Set

$$g_n(t)=\int_0^t f_L(t;\, n)\,dt \qquad (0\leq t\leq 2\pi).$$

The sequence $\{g_n(t)\}$ is uniformly bounded, because

$$|g_n(t)|\leq 2\pi L,$$

and equicontinuous, because

$$|g_n(t'')-g_n(t')|\leq L\,|t''-t'|.$$

Therefore by Arzelà's theorem there exists a subsequence $\{g_{n_i}(t)\}$ uniformly convergent to some function $g(t)$, obviously satisfying the inequality

$$|g(t'')-g(t')|\leq L\,|t''-t'|.$$

From this inequality it follows that

$$g(t)=\int_0^t f(t)\,dt,$$

where $f(t)$ is some measurable function satisfying the inequality

$$-L\leq f(t)\leq L.$$

Since

$$c_k=\int_0^{2\pi} f_L(t;\, n_j)e^{ikt}\,dt=g_{n_j}(2\pi)-ik\int_0^{2\pi} g_{n_j}(t)\,e^{ikt}\,dt=$$

$$=g(2\pi)-ik\int_0^{2\pi} g(t)\,e^{ikt}\,dt=\int_0^{2\pi} f(t)\,e^{ikt}\,dt,$$

$f(t)$ satisfies all the conditions of the theorem.

Remark. If the inequality

$$-L \leq f(t) \leq L$$

is replaced by

$$-L(1-\theta) \leq f(t) \leq L(1+\theta),$$

(the inequality $\alpha \leq f(t) \leq \beta$ may always be written in this form), then inequality (3) in the formulations of Theorems 1 and 2 must be replaced by

$$-2\pi L(1-\theta) \leq c_0 \leq 2\pi L(1+\theta), \tag{3''}$$

and the sequence

$$\gamma, \gamma_1, \cdots, \gamma_{n-1} \qquad (\gamma_0 = \gamma + \bar{\gamma})$$

by the sequence

$$\delta, \delta_1, \delta_2, \cdots, \delta_{n-1}' \qquad (\delta_0 = \delta + \bar{\delta}),$$

where

$$\delta = e^{-\frac{\pi i \theta}{2}} \gamma, \quad \delta_k = e^{-\frac{\pi i \theta}{2}} \gamma_k'' \qquad (k = 1, 2, \cdots, n-1).$$

2. We consider the special case of the L-problem (2) with

$$c_0 = 0, \quad c_k = 2ib_k \qquad (k = 1, 2, 3, \cdots),$$

where the numbers b_k are real.

If $f(t)$ is a solution of L-problem (2) for this case, then by substituting the interval $<-\pi, \pi>$ for the interval $<0, 2\pi>$ and the expression $\frac{1}{2}\{f(t)-f(-t)\}$, which is obviously also a solution, for $f(t)$ we obtain

$$b_k = \int_0^{\pi} f(t) \sin kt\, dt \qquad (k = 1, 2, 3, \cdots) \tag{8}$$

$$-L \leq f(t) \leq L.$$

From this follows

Theorem 3. *In order that there exist a measurable function $f(t)$ satisfying relations* (8), *it is necessary and sufficient that the sequence*

$$\beta_0 = 2, \beta_1, \beta_2, \cdots,$$

defined by the power series

$$\exp\{-\frac{1}{L}(b_1 z + b_2 z^2 + \cdots)\} = 1 + \beta_1 z + \beta_2 z^2 + \cdots$$

be non-negative definite on the circumference.

Theorem 3 may be supplemented by the following proposition:

Theorem 4. *The L-problem*

$$b_k = \int_0^{\pi} f(t) \sin kt\, dt \qquad (k = 1, 2, \cdots, n-1) \tag{8'}$$
$$-L \leq f(t) \leq L$$

has exactly one solution if and only if it admits as a solution an L-function of order $p - 1 < n - 1$. This case is characterized by the fact that the sequence

$$\beta_0 = 2, \ \beta_1, \beta_2, \cdots, \beta_{n-1} \tag{8''}$$

has rank $p < n$.

Proof. We shall assume the sequence (8″) has rank $p < n$. By Theorem 16 of Chapter I there exists a uniquely defined representation

$$\beta_k = \sum_{j=1}^{m} \rho_j \cos kt_j \qquad (k = 0, 1, \cdots, p-1),$$

with $\rho_j > 0$ and

$$0 \leq t_1 < t_2 < \cdots < t_m \leq \pi,$$

and also

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m = p,$$

where $\epsilon_j = 2$, if $0 < t_j < \pi$ and $\epsilon_j = 1$, if $t_j = 0$ or $t_j = \pi$.

From this we obtain a rational function

$$R(z) = \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{1 - z^2}{1 - 2z \cos t_j + z^2} = 1 + \beta_1 z + \beta_2 z^2 + \cdots.$$

Noting that $R(e^{it})$ is an odd function of t whose zeros alternate with its poles, we find after a simple calculation that in the interval $0 < t < \pi$ the sum of the number of zeros and poles of this function always equals $p - 1$. But since these zeros and poles are points of discontinuity in the interval $0 < t < \pi$ of the function

$$f_L(t) = L \operatorname{sign} \{ \frac{1}{i} R(e^{-it}) \},$$

which by Theorem 1 is a solution of the L-problem (8′), the L-problem (8′) has as a solution some L-function of order $p - 1$.

By the note to Theorem 1, this is the unique solution of the L-problem (8′).

Conversely, if the L-problem (8′) has only one solution, then the rank of the sequence (8″) must be less than n, because otherwise there exists an infinite set of non-negative definite extended sequences

$$\beta_0 = 2,\ \beta_1, \cdots, \beta_{n-1}, \beta_n,$$

each of which has its own solution of problem (8′).

3. With the help of the previous two theorems we may easily obtain criteria for the solvability of the exponential L-problem for finite intervals, i.e., criteria for the solvability of the problem

$$s_k = \int_{-1}^{1} u^k g(u)du,\ -L \le g(u) \le L \qquad (k = 0, 1, 2, \cdots). \tag{9}$$

Indeed, setting

$$u = \cos t,\ g(u) = -f(t),$$

we find by (9) that

$$s_k = -\int_0^{\pi} \cos^k t \cdot \sin t \cdot f(t)\,dt,\ -L \le f(t) \le L \qquad (k = 0, 1, 2, \cdots), \tag{9′}$$

and the last equations are equivalent to the equations

$$b_k = -\int_0^{\pi} \sin kt \cdot f(t)\,dt \qquad (k = 1, 2, \cdots), \tag{9″}$$

so that problem (9) is reduced to problem (8), where the sequence $\{b_k\}$ easily may be constructed from the sequence $\{s_k\}$.

This connection between the sequences $\{s_k\}$ and $\{b_k\}$ is conveniently represented in the form of the following identity:

$$\frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \ldots = 2\{b_1 z + b_2 z^2 + b_3 z^3 + \ldots\} \quad \left(x = \frac{1+z^2}{2z}\right). \tag{10}$$

In order to prove identity (10) we note that on the one hand by virtue of (9)

$$\frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \ldots = \int_{-1}^{1} \frac{g(u)\,du}{x-u} = -2\int_0^{\pi} \frac{z \sin t}{1 - 2z\cos t + z^2} f(t)\,dt,$$

and on the other hand by (9″)

$$\int_0^{\pi} \frac{z \sin t}{1 - 2z\cos t + z^2} f(t)\,dt = \sum_{k=1}^{\infty} z^k \int_0^{\pi} \sin kt \cdot f(t) \cdot dt = -\{b_1 z + b_2 z^2 + b_3 z^3 + \ldots\}.$$

Thus by Theorems 3 and 4, problem (9) has a solution if and only if the sequence

$$\beta_0 = 2,\ \beta_1, \beta_2, \cdots$$

is non-negative definite on the circumference, where

$$\exp\left\{\frac{1}{L}(b_1 z + b_2 z^2 + \cdots)\right\} = 1 + \beta_1 z + \beta_2 z^2 + \cdots. \tag{10'}$$

In order to put our results in final form, we set

$$\sqrt{x^2-1}\left\{\frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \frac{\sigma_2}{x^3} + \ldots\right\} = 1 + \beta_1 z + \beta_2 z^2 + \cdots \qquad \left(\sigma_0 = 1,\ x = \frac{1+z^2}{2z}\right) \tag{10''}$$

and find immediately the connection between the sequences $\{s_k\}$ and $\{\sigma_k\}$.

We have by (10''), (10') and (10)

$$\frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \frac{\sigma_2}{x^3} + \ldots = \frac{1}{\sqrt{x^2-1}}\left\{1 + \beta_1 z + \beta_2 z^2 + \ldots\right\} =$$
$$= \frac{1}{\sqrt{x^2-1}} \exp\left\{\frac{1}{L}(b_1 z + b_2 z^2 + \ldots)\right\} =$$
$$= \frac{1}{\sqrt{x^2-1}} \exp\left\{\frac{1}{2L}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \ldots\right)\right\}.$$

On the basis of Theorem 17 of Chapter I and the remark to this theorem, the real sequence $2, \beta_1, \beta_2, \cdots$ is non-negative definite on the circumference if and only if the sequence $\sigma_0, \sigma_1, \sigma_2, \cdots$ is non-negative definite in the interval $< -1, 1 >$. Therefore from Theorems 3 and 4 of this chapter there follows

Theorem 5. *In order that there exist a measurable function* $g(u)$ $(-1 \le u \le 1)$ *satisfying conditions* (9) *it is necessary and sufficient that the sequence*

$$\sigma_0, \sigma_1, \sigma_2, \cdots, \tag{11}$$

defined with the help of the expansion

$$\frac{1}{\sqrt{x^2-1}} \exp\left\{\frac{1}{2L}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \ldots\right)\right\} = \frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \frac{\sigma_2}{x^3} + \ldots \quad (\sigma_0 = 1) \tag{11'}$$

be non-negative definite in the interval $< -1, 1 >$.

The L-moment problem

$$s_k = \int_{-1}^{+1} u^k g(u)\,du, \quad -L \le g(u) \le L \qquad (k = 0, 1, \cdots, n-1)$$

has only one solution if and only if it admits as a solution an L-function of order $p - 1 < n$. *This case is characterized by the fact that the sequence*

$$\sigma_0 = 1,\ \sigma_1, \sigma_2, \cdots, \sigma_n$$

has rank $p < n + 1$.

Remark. If we replaced in conditions (9) the inequality

$$-L \leq g(u) \leq L$$

by the inequality

$$-L(1-\theta) \leq g(u) \leq L(1+\theta),$$

then the sequence (11) would be defined by the expansion

$$\frac{1}{\sqrt{x^2-1}}\left(\frac{x-1}{x+1}\right)^{\frac{\theta}{2}} \exp\left\{\frac{1}{2L}\left(\frac{s_0}{x}+\frac{s_1}{x^2}+\ldots\right)\right\} = \frac{\sigma_0}{x}+\frac{\sigma_1}{x^2}+\frac{\sigma_2}{x^3}+\ldots \quad (\sigma_0 = 1), \tag{11''}$$

and this is the only change which would be necessary in Theorem 5 for this case.

If moreover, we substituted the interval $< a, b >$ for the interval $< -1, 1>$ we would have to determine the sequence (11) from the expansion

$$\frac{1}{\sqrt{(x-a)(x-b)}}\left(\frac{x-b}{x-a}\right)^{\frac{\theta}{2}} \exp\left\{\frac{1}{2L}\left(\frac{s_0}{x}+\frac{s_1}{x^2}+\ldots\right)\right\} = \frac{\sigma_0}{x}+\frac{\sigma_1}{x^2}+\frac{\sigma_2}{x^3}+\ldots \quad (\sigma_0 = 1) \tag{11'''}$$

and require it to be non-negative definite in the interval $< a, b >$.

We note also that one can prove Theorem 5 immediately, and independently of Theorems 3 and 4 (viz. N. Ahiezer and M. Kreĭn [1d]), as the reader can easily see from the proof of the final assertion of Theorem 1.

L. V. Kantorovič [6] proposed another solution to the exponential L-problem. This solution appears correct at first glance but upon closer inspection turns out to be invalid. The conditions cited there are necessary but not sufficient.

We note also that recently Verblunsky wrote a series of papers related to the question under consideration here, apparently unaware of the old work of A. Markov and of the authors' work. Without saying that all Verblunsky's results are contained in the first article of the authors on the L-problem they are [1g] of a very incomplete character.

§2

1. We turn now to the L-problem in the infinite interval $(-\infty, \infty)$.

Theorem 6. *In order that there exist a piecewise continuous function $f(u)$ satisfying the conditions*

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\,du \qquad (k = 0, 1, 2, \cdots, 2m;\ s_0 > 0)$$

$$0 \le f(u) < L, \tag{12}$$

it is necessary and sufficient that the sequence

$$t_0, t_1, \cdots, t_{2m},$$

defined by the expansion

$$e^{\frac{1}{L}\left(\frac{s_0}{x}+\frac{s_1}{x^2}+\cdots+\frac{s_{2m}}{x^{2m+1}}\right)} = 1 + \frac{t_0}{x} + \frac{t_1}{x^2} + \cdots + \frac{t_{2m}}{x^{2m+1}} + \cdots \tag{13}$$

be positive definite in the interval $(-\infty, \infty)$.

Proof. We note at the outset that by relations (13) the quantity $t_k = t_k(L)$ is a polynomial of degree $k+1$ in $\frac{1}{L}$, given by the formulas

$$t_0(L) = \frac{s_0}{L}, \quad t_1(L) = \frac{s_1}{L} + \frac{s_0^2}{2L^2}, \quad t_2(L) = \frac{s_2}{L} + \frac{s_1 s_0}{L^2} + \frac{s_0^2}{6L^3}$$

$$t_k(L) = \frac{1}{(k+1)!\,L^{k+1}} \begin{vmatrix} s_0 & -L & 0 & \cdots & 0 & 0 \\ 2s_1 & s_0 & -2L & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ ks_{k-1} & (k-1)s_{k-2} & (k-2)s_{k-3} & \cdots & s_0 & -kL \\ (k+1)s_k & ks_{k-1} & (k-1)s_{k-2} & \cdots & 2s_1 & s_0 \end{vmatrix} =$$

$$= \frac{s_k}{L} + \frac{s_0 s_{k-1}}{L^2} + \cdots \quad (k > 1). \tag{13'}$$

We shall use these relations later.

In order to prove the sufficiency of the condition of the theorem we take a number L' satisfying the inequality $0 < L' < L$ so that the sequence below is positive definite in the interval $(-\infty, \infty)$

$$t_0(L'), t_1(L'), \cdots, t_{2m}(L');$$

this is possible because the form

$$\sum_0^m t_{i+k}(M) x_i x_k$$

is a continuous function of $M > 0$.

Having thus selected the number L' we shall use Theorem 3 of Chapter I and take the canonical representation

$$t_k(L') = \sum_{i=1}^{m+1} \rho_i \xi_i^k \qquad (k = 0, 1, 2, \cdots, 2m),$$

where $\rho_i > 0$ and $-\infty < \xi_1 < \xi_2 < \cdots < \xi_{m+1} < \infty$.

Thus,

$$1 + \frac{t_0(L')}{x} + \frac{t_1(L')}{x^2} + \cdots + \frac{t_{2m}(L')}{x^{2m+1}} + \cdots = 1 + \sum_{i=1}^{m+1} \frac{\rho_i}{x - \xi_i} = \frac{\psi_{m+1}(x)}{\phi_{m+1}(x)},$$

where

$$\phi_{m+1}(x) = (x-\xi_1)(x-\xi_2)\cdots(x-\xi_{m+1}),$$

$$\psi_{m+1}(x) = (x-\eta_1)(x-\eta_2)\cdots(x-\eta_{m+1}),$$

and as is easily seen

$$\eta_1 < \xi_1 < \eta_2 < \xi_2 < \cdots < \eta_{m+1} < \xi_{m+1}.$$

Setting

$$f^*_{L'}(u) = \frac{L'}{2}\left\{1 - \operatorname{sign}\frac{\psi_{m+1}(u)}{\phi_{m+1}(u)}\right\},$$

we find without difficulty that

$$\int_{-\infty}^{\infty} \frac{f^*_{L'}(u)}{x-u}\,du = L' \log \frac{\psi_{m+1}(x)}{\phi_{m+1}(x)}.$$

Taking the logarithm of the relation (13) after substitution of L' for L, we obtain

$$\frac{s_0}{x} + \frac{s_1}{x^2} + \cdots + \frac{s_{2m}}{x^{2m+1}} + \cdots = L' \log\left\{1 + \frac{t_0(L')}{x} + \frac{t_1(L')}{x^2} + \cdots + \frac{t_{2m}(L')}{x^{2m+1}} + \cdots\right\}.$$

Therefore,

$$\frac{s_0}{x} + \frac{s_1}{x^2} + \cdots + \frac{s_{2m}}{x^{2m+1}} + \cdots = L' \log \frac{\psi_{m+1}(x)}{\phi_{m+1}(x)} = \int_{-\infty}^{\infty} \frac{f^*_{L'}(u)}{x-u}\,du =$$

$$= \sum_{k=0}^{\infty} \frac{1}{x^{k+1}} \int_{-\infty}^{\infty} u^k f^*_{L'}(u)\,du,$$

from which it follows that

$$s_k = \int_{-\infty}^{\infty} u^k f^*_{L'}(u)\,du \qquad (k = 0, 1, \cdots, 2m).$$

We shall now prove the necessity of the condition of the theorem.

Let $f(u)$ satisfy the relations (12). Having selected an arbitrary positive number N, we set

$$s_k^{(N)} = \int_{-N}^{N} u^k f(u)\,du \qquad (k = 0, 1, 2, \cdots, 2m)$$

and define the corresponding quantities $t_k^{(N)} = t_k^{(N)}(L)$ $(k = 0, 1, \cdots, 2m)$.

Taking into consideration the remark to Theorem 5, we find that the sequence

$$\sigma_0, \sigma_1, \cdots, \sigma_{2m+1},$$

which is defined by the expansion

$$\frac{1}{x+N}\exp\left\{\frac{1}{L}\left(\frac{s_0^{(N)}}{x}+\frac{s_1^{(N)}}{x^2}+\cdots+\frac{s_{2m}^{(N)}}{x^{2m+1}}\right)\right\} = \frac{\sigma_0}{x}+\frac{\sigma_1}{x^2}+\cdots+\frac{\sigma_{2m+1}}{x^{2m+2}}+\cdots \qquad (\sigma_0 = 1)$$

is non-negative definite in the interval $(-N, N)$, and thus the forms

$$\sum_0^m (N\sigma_{i+k} \pm \sigma_{i+k+1})x_i x_k$$

are non-negative.

Noting that

$$t_k^{(N)} = N\sigma_k + \sigma_{k+1} \qquad (k = 0, 1, \cdots, 2m),$$

we thus get that the form

$$\sum_0^m t_{i+k}^{(N)} x_i x_k$$

is non-negative for any $N > 0$, and since $\lim_{N\to\infty} t_k^{(N)} = t_k$, the form

$$\sum_0^m t_{i+k} x_i x_k$$

is non-negative, which means that the sequence

$$t_0, t_1, \cdots, t_{2m}$$

is non-negative definite in the interval $(-\infty, \infty)$.

It remains to establish that this sequence is positive definite in the interval $(-\infty, \infty)$.

Assuming otherwise, we find by Theorem 5 of Chapter I, that there is a representation

$$t_k = \sum_{i=1}^{p} \rho_i \xi_i^k \qquad (k = 0, 1, \cdots, 2m-1),$$

$$t_{2m} = \sum_{i=1}^{p} \rho_i \xi_i^{2m} + M \qquad (M \geq 0,\ p \leq m)$$

($p \geq 1$ since $s_0 > 0$), where M certainly equals zero for $p = m$.

Reasoning as in the proof of the sufficiency of the condition of the present theorem we find that

$$s_k = \int_{-\infty}^{\infty} u^k f_L^*(u)\, du \qquad (k = 0, 1, \cdots, 2m-1), \tag{14'}$$

where

$$f_L^*(u) = \frac{L}{2}\left\{1 - \operatorname{sign}\frac{\psi_p(u)}{\phi_p(u)}\right\}$$

and also in the case $p = m$ we have the equality

$$s_{2m} = \int_{-\infty}^{\infty} u^{2m} f_L^*(u)\, du. \tag{14''}$$

Because the condition

$$0 \leq f(u) < L,$$

we have

$$\operatorname{sign}\{f_L^*(u) - f(u)\} \cdot \operatorname{sign}\{\phi_p(u)\psi_p(u)\} \leq 0.$$

On the other hand, for $p < m$ as well as for $p = m$, by (14') and (14'') we have

$$\int_{-\infty}^{\infty} \{f_L^*(u) - f(u)\}\phi_p(u)\psi_p(u)\, du = 0,$$

from which it follows that $f(u) = f_L^*(u)$ almost everywhere, but by (12) this is impossible, since $f_L^*(u)$ equals L on entire intervals.

2. Recalling the final assertion of Theorem 3 of Chapter I, it is easily seen that whatever the real number s_{2m+1} may be for the sequence $t_0, t_1, \cdots, t_{2m}$ positive definite in the interval $(-\infty, \infty)$, there exists a function $f(u)$ satisfying conditions (12) and the condition

$$s_{2m+1} = \int_{-\infty}^{\infty} u^{2m+1} f(u)\, du. \tag{12'}$$

Moreover, the proof of Theorem 6 shows that it is necessary that the infinite sequence

$$t_0(L), t_1(L), t_2(L), \cdots \tag{15}$$

be non-negative definite in the interval $(-\infty, \infty)$ in order that there exist a function $f(u)$ which will satisfy the inequality

$$0 \leq f(u) \leq L \tag{16'}$$

and the relations

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\, du \tag{16''}$$

for all integers $k \geq 0$.

We shall show the converse is also valid. If an infinite sequence (15) is non-negative definite in the interval $(-\infty, \infty)$, but not positive and p is the order of the last determinant different from zero in the sequence

$$\Delta_0(L) = t_0(L), \quad \Delta_1(L) = \begin{vmatrix} t_0(L) & t_1(L) \\ t_1(L) & t_2(L) \end{vmatrix}, \ldots, \Delta_k(L) = |t_{i+j}(L)|_0^k, \ldots,$$

then the canonical representation

$$t_k(L) = \sum_{i=1}^{p} \rho_i \xi_i^k \qquad (k = 0, 1, 2, \cdots, 2p-1)$$

is valid also for $k = 2p, 2p+1, \cdots$ and hence the end of the proof of Theorem 6 shows that in this case there does exist a function $f(u)$ satisfying inequality (16') and the conditions (16'') for $k = 0, 1, 2, \cdots$ which is equal to the function

$$f_L^*(u) = \frac{L}{2}\left\{1 - \operatorname{sign}\frac{\psi_p(u)}{\phi_p(u)}\right\}$$

almost everywhere, takes on both of the values 0, L and has $2p$ points of discontinuity.

Let us now assume that the infinite sequence (15) is positive definite in the interval $(-\infty, \infty)$.

From Theorem 6 for any natural number n there exists a function $f(u; n)$ satisfying the conditions

$$s_k = \int_{-\infty}^{\infty} u^k f(u; n)\, du, \quad 0 \leq f(u; n) < L \qquad (k = 0, 1, \cdots, 2n).$$

Setting

$$\int_{-\infty}^{u} f(u; n)\, du = \Phi(u; n) \qquad (n = 1, 2, \cdots),$$

so that $\Phi(u; n)$ is a non-decreasing function satisfying the inequalities

$$\begin{aligned} &0 \leqslant \Phi(u; n) \leqslant \int_{-\infty}^{\infty} f(u; n)\, du = s_0, \qquad (u'' > u') \\ &0 \leqslant \Phi(u''; n) - \Phi(u'; n) < L(u'' - u') \qquad (n = 1, 2, 3, \ldots) \end{aligned} \tag{17}$$

and applying Helly's theorem as in the proof of Theorem 7 of Chapter II, we find a certain non-decreasing function $\Phi(u)$ which satisfies the relations

$$s_k = \int_{-\infty}^{\infty} u^k d\,\Phi(u) \qquad (k = 0, 1, 2, \cdots)$$

as well as

$$\lim_{i \to \infty} \Phi(u; n_i) = \Phi(u)$$

at all points where the function $\Phi(u)$ is continuous.

From inequalities (17) it follows that $\Phi(u)$ is continuous everywhere and satisfies the Cauchy-Lipschitz condition

$$0 \leq \Phi(u'') - \Phi(u') \leq L(u'' - u') \qquad (u'' > u').$$

Therefore by the theorem of Lebesgue,

$$\Phi(u) = \int_{-\infty}^{u} f(u)\,du,$$

where $f(u)$ is some measurable function which satisfies inequality (16′).

Our assertion is proved and thus we have

Theorem 7. *For the existence of a function $f(u)$ satisfying the inequality*

$$0 \leq f(u) \leq L$$

for which

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\,du \qquad (k = 0, 1, 2, \cdots),$$

it is necessary and sufficient that the sequence

$$t_0(L), t_1(L), t_2(L), \cdots$$

is non-negative definite in the interval $(-\infty, \infty)$.

In this connection, if this non-negative definite sequence is not positive definite and p is the order of the last determinant different from zero in the sequence

$$\Delta_0(L) = t_0(L), \quad \Delta_1(L) = \begin{vmatrix} t_0(L) & t_1(L) \\ t_1(L) & t_2(L) \end{vmatrix}, \cdots,$$

then the function $f(u)$ is essentially unique and almost everywhere equal to the function

$$f_L^*(u) = \frac{L}{2}\left\{1 - \operatorname{sign}\frac{\psi_p(u)}{\phi_p(u)}\right\},$$

where $\phi_p(u)$ and $\psi_p(u)$ are certain polynomials of degree p with alternating roots which may be determined from the expansion

$$\frac{\psi_p(x)}{\phi_p(x)} = 1 + \frac{t_0(L)}{x} + \frac{t_1(L)}{x^2} + \cdots + \frac{t_{2p-1}(L)}{x^{2p}} + \cdots.$$

§3

1. The L-problem considered in the previous paragraph obviously has no solution if the sequence $\{s_k\}$ $(s_0 > 0)$ is not positive definite in the interval $(-\infty, \infty)$, since in this case if there exists a non-decreasing function $\sigma(u)$ satisfying the conditions

$$s_k = \int_{-\infty}^{\infty} u^k d\sigma(u) \qquad (k = 0, 1, 2, \cdots),$$

then this function is necessarily piecewise constant, i.e., discontinuous, which means that the condition

$$\sigma(u'') - \sigma(u') \leq L(u'' - u') \qquad (u'' > u')$$

is not fulfilled.

Therefore we naturally come to the following question: Given that a sequence $s_0, s_1, \cdots, s_n$ is positive definite in the interval $(-\infty, \infty)$, do there exist non-negative bounded functions $f(u)$ for which

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\, du \qquad (k = 0, 1, 2, \cdots, n),$$

and if they exist, what is the value for these functions of

$$\inf\{\text{vrai} \max_{-\infty < u < \infty} f(u)\}?$$

To answer this question we must study the quantities $t_k = t_k(L)$ introduced above and the determinants $\Delta_k(L)$ formed from them, as functions of L.

Lemma 1. *The determinants $\Delta_k(L)$ satisfy the relations*

$$\Delta_k(L) = (-1)^{k+1}\Delta_k(-L).$$

Proof. Recalling the definition of the quantities $t_k(L)$ we note that

$$\frac{1}{u-v}\left\{e^{\frac{1}{L}(s_0u + s_1u^2 + \cdots)} - e^{\frac{1}{L}(s_0v + s_1v^2 + \cdots)}\right\} = \sum_{i,k=0}^{n} t_{i+k}(L)u^i v^k + \cdots.$$

Multiplying both sides of this equation by

$$e^{-\frac{1}{2L}(s_0u+s_1u^2+\cdots)} = 1 + t_0(-2L)u + t_1(-2L)u^2 + \cdots = A_0 + A_1u + \cdots$$

and

$$e^{-\frac{1}{2L}(s_0v+s_1v^2+\cdots)} = 1 + t_0(-2L)v + t_1(-2L)v^2 + \cdots = A_0 + A_1v + \cdots$$

we obtain

$$\frac{1}{2}\sum_{i,k=0}^{n} c_{i,k}(L)u^iv^k + \cdots = \frac{\text{sin hyp}\left\{\frac{s_0u+s_1u^2+\cdots}{2L} - \frac{s_0v+s_1v^2+\cdots}{2L}\right\}}{u-v} =$$

$$= \frac{1}{2}\sum_{i,k=0}^{n} t_{i+k}(L)(A_0u^i + A_1u^{i+1} + \cdots + A_{n-i}u^n)(A_0v^k +$$

$$+ A_1v^{k+1} + \cdots + A_{n-k}v^n) + \cdots.$$

Therefore

$$\sum_{i,k=0}^{n} c_{i,k}(L)u_iv_k = \sum_{i,k=0}^{n} t_{i+k}(L)\xi_i\eta_k,$$

where

$$\begin{aligned}
\xi_0 &= A_0u_0 + A_1u_1 + \cdots + A_nu_n,\\
\xi_1 &= \quad A_0u_1 + \cdots + A_{n-1}u_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\xi_n &= \quad A_0u_n;\\
\eta_0 &= A_0v_0 + A_1v_1 + \cdots + A_nv_n,\\
\eta_1 &= \quad A_0v_1 + \cdots + A_{n-1}v_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\eta_n &= \quad A_0v_n.
\end{aligned}$$

Since the determinant of this transformation is $1 (d = A_0^n = 1)$, we have

$$\Delta_n(L) = \begin{vmatrix} c_{0,0}(L)\, c_{0,1}(L) \ldots c_{0,n}(L) \\ c_{1,0}(L)\, c_{1,1}(L) \ldots c_{1,n}(L) \\ \cdots\cdots\cdots\cdots\cdots \\ c_{n,0}(L)\, c_{n,1}(L) \ldots c_{n,n}(L) \end{vmatrix}$$

and our assertion follows since $c_{i,k}(L)$ is an odd function of L.

We have been dealing repeatedly with a pair of polynomials $\phi_p(x)$ and $\psi_p(x)$

of degree p which satisfy the relation

$$\frac{\psi_p(x)}{\phi_p(x)} = 1 + \frac{t_0(L)}{x} + \cdots + \frac{t_{2p-1}(L)}{x^{2p}} + \cdots, \tag{18}$$

where

$$\Delta_k(L) > 0 \qquad (k = 0, 1, \cdots, p-1;\ L > 0).$$

We shall show that

$$\frac{\psi_p(x)}{\phi_p(x)} = (-1)^p \frac{\Delta_p(x; -L)}{\Delta_p(x; L)}, \tag{18'}$$

where

$$\Delta_k(x; L) = \begin{vmatrix} t_0(L) & t_1(L) \ldots t_k(L) \\ t_1(L) & t_2(L) \ldots t_{k+1}(L) \\ \cdots\cdots\cdots\cdots\cdots \\ t_{k-1}(L) & t_k(L) \ldots t_{2k-1}(L) \\ 1 & x \ \ldots \ x^k \end{vmatrix}. \tag{18''}$$

From the expansion (18) it follows that $\phi_p(x)$ is an orthogonal polynomial of degree p relative to the sequence $t_0(L), \cdots, t_{2p-1}(L)$; and since the coefficient of the highest term of $\phi_p(x)$ is 1,

$$\phi_p(x) = \frac{1}{\Delta_{p-1}(L)} \Delta_p(x; L).$$

Noting that

$$\frac{\phi_p(x)}{\psi_p(x)} = e^{-\frac{1}{L}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \cdots\right)} = 1 + \frac{t_0(-L)}{x} + \cdots + \frac{t_{2p-1}(-L)}{x^{2p}} + \cdots,$$

we obtain by the same considerations that

$$\psi_p(x) = \frac{1}{\Delta_{p-1}(-L)} \Delta_p(x; -L),$$

and it remains to consider the equality

$$\Delta_{p-1}(-L) = (-1)^p \Delta_{p-1}(L).$$

Lemma 2. *If the form*

$$\sum_0^m s_{i+k} x_i x_k \tag{19}$$

is positive, there exists a positive number λ_m *such that the form*

$$\sum_0^m t_{i+k}(L) x_i x_k \tag{20}$$

is positive for $L > \lambda_m$ *and is not non-negative for* $0 < L < \lambda_m$.

This number λ_m *is the largest root of the equation*

$$\Delta_m(L) = 0 \tag{21}$$

and being a function of $s_0, s_1, \cdots, s_{2m}$ *it satisfies the inequality*

$$\lambda_{m-1}(s_0, s_1, \cdots, s_{2m-2}) \leq \lambda_m(s_0, s_1, \cdots, s_{2m}), \tag{22}$$

so that in particular

$$\lambda_m(s_0, s_1, \cdots, s_{2m}) \geq \frac{s_0^2}{2\sqrt{3}\sqrt{s_0 s_2 - s_1^2}} = \lambda_1(s_0, s_1, s_2).$$

Proof. Since by (13′)

$$\lim_{L \to \infty} \sum_0^m L t_{i+k}(L) x_i x_k = \sum_0^m s_{i+k} x_i x_k,$$

the form (20) is positive for sufficiently large $L > 0$; if

$$L < \frac{s_0^2}{2\sqrt{3}\sqrt{s_0 s_2 - s_1^2}},$$

then form (20) is not non-negative, since

$$\Delta_1(L) = \begin{vmatrix} t_0(L) & t_1(L) \\ t_1(L) & t_2(L) \end{vmatrix} = \frac{1}{L^2}\left\{ \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} - \frac{s_0^4}{12L^2} \right\}.$$

Conversely, if the form (20) is positive for $L = \lambda' > 0$, then it will be positive also for $L = \lambda'' > \lambda'$, because from the positiveness of form (20) for $L = \lambda' > 0$ it follows from Theorem 6 that there exists a function $f(u)$ satisfying the relations

$$0 \leq f(u) < \lambda', \quad s_k = \int_{-\infty}^{\infty} u^k f(u)\, du \qquad (k = 0, 1, \cdots, 2m),$$

which means that a fortiori it satisfies the relations

$$0 \leq f(u) < \lambda'', \quad s_k = \int_{-\infty}^{\infty} u^k f(u)\, du \qquad (k = 0, 1, \cdots, 2m).$$

Since at the values of L for which form (20) is non-negative singular the function

$$\Delta_m(L)$$

vanishes, they are therefore isolated, so that the lower bound of those values of L for which form (20) is positive definite is the above number λ_m and it is the

largest root of equation (21).

Inequality (22) is an obvious consequence of Theorem 6.

Theorem 8. *If the form*

$$\sum_{0}^{m} s_{i+k} x_i x_k \tag{19}$$

is positive and s_{2m+1} *is an arbitrary real number, then there exist non-negative functions* $f(u)$ *satisfying the conditions*

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\, du \qquad (k = 0, 1, 2, \cdots, 2m + 1), \tag{23}$$

and for these functions $f(u)$

$$\inf \{ \operatorname*{vrai\,max}_{-\infty < u < \infty} f(u) \} = \lambda_m,$$

where λ_m *is the number defined in Lemma* 2.

Proof. The statement immediately follows from Lemma 2.

The theorem deduced does not claim that there exists a non-negative function $f(u)$ satisfying conditions (23) for which $\operatorname*{vrai\,max}_{-\infty < u < \infty} f(u) = \lambda_m$; we shall show below that such a function need not exist.

However, it is true that

Theorem 9. *In order that there exist a non-negative function* $f(u)$ *satisfying the conditions*

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\, du \qquad (k = 0, 1, \cdots, 2m - 1,\ s_0 > 0) \tag{24'}$$

$$s_{2m} \geq \int_{-\infty}^{\infty} u^{2m} f(u)\, du, \tag{24''}$$

it is necessary and sufficient that the following form be positive:

$$\sum_{0}^{m} s_{i+k} x_i x_k. \tag{19}$$

For non-negative functions $f(u)$ *satisfying conditions* (24') *and* (24'') *the inequality*

$$\operatorname*{vrai\,max}_{-\infty < u < \infty} f(u) \geq \lambda_m$$

is valid, where equality is attained if and only if $f(u)$ *is almost everywhere equal to the function*

$$f^*_{\lambda_m}(u) = \frac{\lambda_m}{2}\left\{1 - \operatorname{sign}\frac{\psi_p(u)}{\phi_p(u)}\right\},$$

where p *is the order of the last determinant different from zero in the sequence*

$$\Delta_0(\lambda_m), \Delta_1(\lambda_m), \cdots, \Delta_{m-1}(\lambda_m);$$

and furthermore, if $p = m$, *then the function* $f^*_{\lambda_m}(u)$ *satisfies the inequality*

$$s_{2m} = \int_{-\infty}^{\infty} u^{2m} f^*_{\lambda_m}(u)\, du. \qquad (24''')$$

Proof. We have no need for the necessity of the condition so we prove sufficiency. Let the form (19) be positive. Because of the form

$$\sum_0^m t_{i+k}(\lambda_m) x_i x_k$$

is non-negative and singular there is a uniquely determined representation

$$t_k(\lambda_m) = \sum_{i=1}^{p} \rho_i \xi_i^k \qquad (k = 0, 1, \cdots, 2m-1),$$

$$t_{2m}(\lambda_m) = \sum_{i=1}^{p} \rho_i \xi_i^{2m} + M,$$

where $p < m + 1$ and $M \geq 0$, and M certainly equals zero if $p = m$.

From this representation it follows that

$$(-1)^p \frac{\Delta_p(x; -\lambda_m)}{\Delta_p(x; \lambda_m)} = 1 + \sum_{i=1}^{p} \frac{\rho_i}{x - \xi_i} = 1 + \frac{t_0(\lambda_m)}{x} + \cdots + \frac{t_{2m}(\lambda_m) - M}{x^{2m+1}} + \cdots.$$

Therefore the function

$$f^*_{\lambda_m}(u) = \frac{\lambda_m}{2}\left\{1 - \operatorname{sign}\frac{(-1)^p \Delta_p(u; -\lambda_m)}{\Delta_p(u; \lambda_m)}\right\}$$

satisfies the equations

$$s_k = \int_{-\infty}^{\infty} u^k f^*_{\lambda_m}(u)\, du \qquad (k = 0, 1, \cdots, 2m-1)$$

$$\sigma_{2m} = \int_{-\infty}^{\infty} u^{2m} f^*_{\lambda_m}(u)\, du,$$

where σ_{2m} is defined by the conditions

$$e^{\frac{1}{\lambda_m}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \cdots + \frac{\sigma_{2m}}{x^{2m+1}}\right)} = 1 + \frac{t_0(\lambda_m)}{x} + \cdots + \frac{t_{2m}(\lambda_m) - M}{x^{2m+1}} + \cdots,$$

so that by (13′)

$$t_{2m}(\lambda_m) - M = \frac{\sigma_{2m}}{\lambda_m} + \frac{s_0 s_{2m-1}}{\lambda_m^2} + \cdots,$$

and

$$t_{2m}(\lambda_m) = \frac{s_{2m}}{\lambda_m} + \frac{s_0 s_{2m-1}}{\lambda_m^2} + \cdots.$$

This means

$$\sigma_{2m} \leq s_{2m},$$

and the equality is certainly attained if $p = m$, that is if $\lambda_m > \lambda_{m-1}$.

Thus, the existence of a function $f(u)$ which satisfies conditions (24′) and (24″) and the inequality

$$0 \leq f(u) \leq \lambda_m$$

is established.

We now suppose that $f(u)$ is some function which satisfies conditions (24′) and (24″) and the inequality

$$0 \leq f(u) \leq L \leq \lambda_m$$

and is not equal to $f^*_{\lambda_m}(u)$ almost everywhere.

Since the difference

$$f^*_{\lambda_m}(u) - f(u)$$

and the polynomials

$$(-1)^{p-1}\Delta_p(u; -\lambda_m)\,\Delta_p(u; \lambda_m)$$

of degree $2p$ have the same sign almost everywhere,

$$(-1)^{p-1}\int_{-\infty}^{\infty} \{f^*_{\lambda_m}(u) - f(u)\}\,\Delta_p(u; -\lambda_m)\,\Delta_p(u; \lambda_m)\,du > 0$$

or

$$\int_{-\infty}^{\infty} \{f^*_{\lambda_m}(u) - f(u)\}\,u^{2p}\,du < 0.$$

But this inequality is absurd as for $p < m$ the left side equals zero and for $p = m$ it is greater than or equal to zero on the basis of (24″) and (24‴).

The theorem is thus completely proved.

3. It will be useful later to introduce the following terminology: We shall say that a sequence

$$s_0, s_1, s_2, \cdots, s_{2m},$$

which is positive definite in the interval $(-\infty, \infty)$ is *complete* if for the function

$f^*(u)$ which minimizes the value of

$$\operatorname*{vrai\,max}_{-\infty<u<\infty} f(u)$$

under the conditions

$$f(u) \geq 0,$$

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\,du \qquad (k = 0, 1, \cdots, 2m-1),$$

$$s_{2m} \geq \int_{-\infty}^{\infty} u^{2m} f(u)\,du,$$

the equality

$$s_{2m} = \int_{-\infty}^{\infty} u^{2m} f^*(u)\,du$$

holds, and incomplete in the opposite case.

From Theorem 9 follows the

Corollary. *If a sequence*

$$s_0, s_1, \cdots, s_{2m}$$

is positive definite and if $\lambda_m > \lambda_{m-1}$ *then this sequence is complete.*

Indeed by the inequality $\lambda_m > \lambda_{m-1}$ none of the determinants

$$\Delta_0(\lambda_m), \Delta_1(\lambda_m), \cdots, \Delta_{m-1}(\lambda_m)$$

is zero, so that the number p of Theorem 9 equals m.

Lemma 3. *If one of the two (positive definite) sequences*

$$s_0, s_1, \cdots, s_{2m}, \tag{25'}$$

$$s_0, s_1, \cdots, s_{2m-2} \tag{25''}$$

is incomplete then the other must be complete.

Proof. Let us assume that the sequence (25′) is incomplete, so that by the corollary to Theorem 9: $\lambda_m = \lambda_{m-1}$. Therefore the function

$$f^*(u) = f^*_{\lambda_m}(u),$$

which satisfies the conditions

$$0 \leq f^*(u) \leq \lambda_m,$$

$$s_k = \int_{-\infty}^{\infty} u^k f^*(u)\,du \qquad (k = 0, 1, \cdots, 2m-1),$$

$$s_{2m} > \int_{-\infty}^{\infty} u^{2m} f^*(u)\,du,$$

also satisfies the inequality

$$0 \leq f^*(u) \leq \lambda_{m-1}$$

and this means the sequence (25″) is complete.

By virtue of what we have already proved, if (25″) is incomplete then the (positive definite) sequence (25′) cannot be incomplete. Thus the lemma is proved.

Let us now assume that we are given the positive definite sequence (25″)

$$s_0, s_1, \cdots, s_{2m-2} \tag{25″}$$

and some real number s_{2m-1}.

Considering the extended sequence

$$s_0, s_1, \cdots, s_{2m-1}, s_{2m} \tag{25′}$$

with the variable number s_{2m} which satisfies the inequality

$$s_{2m} > u_{2m} = u_{2m}(s_0, s_1, \cdots, s_{2m-1}),$$

where u_{2m} is determined from the equation

$$\begin{vmatrix} s_0 & s_1 & \cdots & s_m \\ s_1 & s_2 & \cdots & s_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_{m-1} & s_m & \cdots & s_{2m-1} \\ s_m & s_{m+1} & \cdots & u_{2m} \end{vmatrix} = 0,$$

we consider $\lambda_m = \lambda_m(s_{2m})$ as a function of s_{2m}.

By the definition of λ_m, it is apparent that $\lambda_m(s_{2m})$ is a non-increasing function of s_{2m}.

Lemma 4. *If the sequence* (25′) *is incomplete for* $s_{2m} = s'_{2m}$, *then it is incomplete for any* $s_{2m} \geq s'_{2m}$.

Indeed, by the corollary to Theorem 9

$$\lambda_m(s'_{2m}) = \lambda_{m-1}.$$

On the other hand,

$$\lambda_m(s_{2m}) \geq \lambda_{m-1}$$

and

$$\lambda_m(s_{2m}) \leq \lambda_m(s'_{2m}) \quad \text{for} \quad s_{2m} \geq s'_{2m}.$$

Therefore

$$\lambda_m(s_{2m}) = \lambda_{m-1}$$

for any $s_{2m} \geq s'_{2m}$.

This means that the function $f^*(u)$, which minimizes the quantity

$$\text{vrai max}_{-\infty < u < \infty} f(u)$$

under the conditions

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\,du \qquad (k = 0, 1, \cdots, 2m-1),$$

$$s_{2m} \geq \int_{-\infty}^{\infty} u^{2m} f(u)\,du,$$

$$f(u) \geq 0,$$

coincides with $f^*_{\lambda_m - 1}(u)$ and for this function

$$\int_{-\infty}^{\infty} u^{2m} f^*_{\lambda_m - 1}(u)\,du < s'_{2m} \leq s_{2m},$$

which proves our assertion.

From the proof of Lemma 4 there follows

Corollary. *If the sequence* (25′) *is incomplete for* $s_{2m} = s'_{2m}$, *then the function* $\lambda_m(s_{2m})$ *has a constant value* λ_{m-1} *for* $s_{2m} \geq s'_{2m}$.

We shall now demonstrate that the positive definite sequence

$$s_0, s_1, \cdots, s_{2m-1}$$

may always be extended so that the sequence (25′) obtained is complete.

To this end we take some number L^* exceeding λ_{m-1} and determine $s_{2m} = s^*_{2m}$ from the condition

$$\Delta_m(L^*) = 0.$$

Since $\lambda_m(s^*_{2m})$ is the greatest root of the equation

$$\Delta_m(L) = 0,$$

we have

$$\lambda_m(s^*_{2m}) \geq L^* > \lambda_{m-1}.$$

Therefore, by Theorem 7

$$s^*_{2m} > u_{2m},$$

and by the corollary to Theorem 9, the sequence (25′) will be complete for $s_{2m} = s^*_{2m}$.

The facts proved in connection with Lemma 4 show that two cases are possible: either there exists some number $v_{2m} = v_{2m}(s_0, s_1, \cdots, s_{2m-1})$ such that the sequence (25′) is complete for

$$u_{2m} < s_{2m} \leq v_{2m}$$

and incomplete for

$$s_{2m} > v_{2m},$$

or the sequence (25′) is complete for any $s_{2m} > u_{2m}$. Moreover, we may consider the second case as being included in the first for $v_{2m} = \infty$.

The corollary to Lemma 4 says: for $s_{2m} \geq v_{2m}$ the function $\lambda_m(s_{2m})$ maintains a constant value λ_{m-1}.

We shall now prove that for $u_{2m} < s_{2m} \leq v_{2m}$ the function $\lambda_m(s_{2m})$ is strictly decreasing. Since for $u_{2m} < s'_{2m} < s''_{2m} \leq v_{2m}$ it is true that

$$\lambda_m(s'_{2m}) \geq \lambda_m(s''_{2m}),$$

it is enough for us to prove that the following equality is impossible:

$$\lambda_m(s'_{2m}) = \lambda_m(s''_{2m}). \tag{26}$$

Assume the opposite and denote by $f^*(u)$ that function which minimizes the quantity

$$\operatorname*{vrai\,max}_{-\infty < u < \infty} f(u) \tag{27}$$

for the conditions

$$f(u) \geq 0, \tag{27′}$$

$$s_k = \int_{-\infty}^{\infty} u^k f(u)\,du \qquad (k = 0, 1, \cdots, 2m-1), \tag{27″}$$

$$s'_{2m} = \int_{-\infty}^{\infty} u^{2m} f(u)\,du.$$

By (26) and Theorem 9, $f^*(u)$ is also that unique function (up to a set of measure zero) which minimizes the quantity (27) under the conditions (27′), (27″), and

$$s''_{2m} \geq \int_{-\infty}^{\infty} u^{2m} f(u)\,du.$$

But this means that the sequence

$$s_0, s_1, \cdots, s_{2m-1}, s''_{2m}$$

is incomplete, which is impossible because

$$s''_{2m} \leq v_{2m}.$$

Thus we have proved the following proposition:

Theorem 10. *If the sequence*

$$s_0, s_1, \cdots, s_{2m-2}$$

is positive definite in the interval $(-\infty, \infty)$ *and* s_{2m-1} *is an arbitrary real number, then there exist two numbers*

$$u_{2m} = u_{2m}(s_0, s_1, \cdots, s_{2m-1}), \quad v_{2m} = v_{2m}(s_0, s_1, \cdots, s_{2m-1}) \leq \infty$$

such that the sequence

$$s_0, s_1, \cdots, s_{2m}$$

is complete for

$$u_{2m} < s_{2m} \leq v_{2m}$$

and incomplete for

$$s'_{2m} > v_{2m}.$$

It remains for us to state the rule for finding the value v_{2m}. To this end we recall formula (13′)

$$t_{2m}(L) = \frac{s_{2m}}{L} + \frac{s_0 s_{2m-1}}{L^2} + \cdots = \frac{s_{2m}}{L} + \Phi(L; s_0, s_1, \cdots, s_{2m-1}).$$

Since for $s_{2m} = v_{2m}$ it is true that

$$\lambda_m = \lambda_{m-1},$$

we obtain for v_{2m} the expression

$$v_{2m} = -\lambda_{m-1} \cdot \Phi(\lambda_{m-1};\ s_0,\ s_1, \cdots,\ s_{2m-1}) -$$

$$- \lambda_{m-1} \lim_{L \to \lambda_{m-1}} \frac{\begin{vmatrix} t_0(L) & \cdots t_m(L) \\ \cdots & \cdots \\ t_{m-1}(L) \cdots t_{2m-1}(L) \\ t_m(L) & \cdots \quad 0 \end{vmatrix}}{\Delta_{m-1}(L)}. \tag{28}$$

Example 1. Let

$$s_0 = 1, \ s_1 = 0, \ s_2 = 1.$$

With the aid of formula (13′) we find that

$$t_0(L) = \frac{1}{L},\ t_1(L) = \frac{1}{2L^2},\ t_2(L) = \frac{1}{L} + \frac{1}{6L^3},$$

$$t_3(L) = \frac{s_3}{L} + \frac{1}{L^2} + \frac{1}{24L^4},\ t_4(L) = \frac{s_4}{L} + \frac{s_3}{L^2} + \frac{1}{2L^3} + \frac{1}{120L^5}.$$

Thus,

$$\Delta_1(L) = \begin{vmatrix} t_0(L) t_1(L) \\ t_1(L) t_2(L) \end{vmatrix} = \frac{1}{L^2}\left(1 - \frac{1}{12L^2}\right),$$

so that

$$\lambda_1 = \frac{1}{2\sqrt{3}}.$$

Therefore by (28)

$$v_4 = \frac{9}{5} + \lim_{L \to \frac{1}{2\sqrt{3}}} \frac{s_3^2}{1 - \frac{1}{12L^2}};$$

thus v_4 is finite only for $s_3 = 0$ and in that case is exactly equal to $\frac{9}{5}$.

On the other hand

$$u_4 = 1 + s_3^2.$$

Example 2. We shall show that one may construct an infinite sequence

$$s_0, s_1, s_2, \cdots$$

such that the sequence

$$s_0, s_1, \cdots, s_{2m}$$

is complete for $m = 2k - 1$ and incomplete for $m = 2k$ $(k = 1, 2, 3, \cdots)$. The sequence

$$s_0 = 1, s_1 = 0, s_2 = 1$$

is complete and the sequence

$$s_0 = 1,\ s_1 = 0,\ s_2 = 1,\ s_3 = 0,\ s_4 = 2,$$

in view of the previous example, is incomplete.

Suppose now that the sequence

$$s_0, s_1, \cdots, s_{4n} \qquad (n \geq 1)$$

is already constructed and satisfies the requirements stated, so that it is incomplete.

If in an arbitrary manner we choose numbers s_{4n+1} and s_{4n+2} which satisfy the inequality

$$\begin{vmatrix} s_0 & \cdots & s_{2n+1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ s_{2n+1} & \cdots & s_{4n+2} \end{vmatrix} > 0,$$

we obtain in view of Lemma 3 the complete sequence

$$s_0, s_1, \cdots, s_{4n+2}$$

and find a function $f^*(u)$ satisfying the relations

$$s_k = \int_{-\infty}^{\infty} u^k f^*(u)\,du \qquad (k = 0, 1, \cdots, 4n+2),$$

$$0 \le f^*(u) \le \lambda_{2n+1}.$$

Setting

$$s_{4n+3} = \int_{-\infty}^{\infty} u^{4n+3} f^*(u)\,du,$$

$$s_{4n+4} > \int_{-\infty}^{\infty} u^{4n+4} f^*(u)\,du,$$

we obtain the obviously incomplete sequence

$$s_0, s_1, \cdots, s_{4n+4}.$$

It remains to apply induction.

§4

1. The questions with which we were concerned in the previous paragraph in connection with the exponential L-problem for infinite intervals arise also in connection with other L-problems we have studied (trigonometric and exponential for finite intervals) and even allow a simpler solution, since we no longer need to distinguish the cases when the sequence is complete and incomplete.

Let us consider for example the exponential L-problem

$$s_k = \int_{-1}^{1} u^k f(u)\,du \qquad (k = 0, 1, \cdots, n), \tag{29'}$$

$$-L(1-\theta) \le f(u) \le L(1+\theta) \qquad (-1 \le \theta \le 1). \tag{29''}$$

With no assumption about the real numbers

$$s_0, s_1, s_2, \cdots, s_n, \tag{29'''}$$

let there be given a number θ satisfying the inequality $-1 < \theta < 1$, and suppose that the sequence (29‴) is positive definite (or non-negative definite) in the interval $<-1, 1>$, if $\theta = 1$ (or $\theta = -1$). We shall prove to begin with that the problems (29′) and (29″) are always solvable for sufficiently large $L > 0$.

We suppose initially that $-1 < \theta < 1$. Setting

$$\frac{1}{\sqrt{x^2-1}}\left(\frac{x-1}{x+1}\right)^{\frac{\theta}{2}} = \frac{\sigma_0'}{x} + \frac{\sigma_1'}{x^2} + \ldots + \frac{\sigma_{n+1}'}{x^{n+2}} + \ldots \qquad (\sigma_0' = 1)$$

and noting that the left member of this equation is

$$\frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\left(\frac{1-u}{1+u}\right)^{\frac{\theta}{2}}\frac{du}{x-u} = \int_{-1}^{1}\frac{d\omega(u)}{x-u},$$

where

$$\omega(u) = \frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{u}\frac{1}{\sqrt{1-u^2}}\left(\frac{1-u}{1+u}\right)^{\frac{\theta}{2}}du,$$

so that $\omega(u)$ is an increasing function, we conclude that the sequence

$$\sigma_0', \sigma_1', \ldots, \sigma_{n+1}'$$

is positive definite in the interval $<-1, 1>$.

Therefore, for sufficiently large $L > 0$ the sequence

$$\sigma_0(L), \sigma_1(L), \ldots, \sigma_{n+1}(L),$$

which is defined by the expansion

$$\frac{1}{\sqrt{x^2-1}}\left(\frac{x-1}{x+1}\right)^{\frac{\theta}{2}}\exp\left\{\frac{1}{2L}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \ldots + \frac{s_n}{x^{n+1}}\right)\right\} =$$

$$= \frac{\sigma_0}{x} + \frac{\sigma_1(L)}{x^2} + \ldots \quad (\sigma_0 = 1) \qquad (11'')$$

will be positive definite in the interval $<-1, 1>$, which means on the basis of the note to Theorem 5 that problem (29′), (29″) is solvable for sufficiently large $L > 0$.

We now suppose that $|\theta| = 1$; for the sake of definiteness, let $\theta = 1$. We are given that the sequence (29‴) is positive definite in the interval $<-1, 1>$; therefore there exists a representation

$$s_k = \sum_{i=1}^{m}\rho_i\xi_i^k \qquad \left(k = 0, 1, \ldots, n;\ m = \left[\frac{n}{2}\right] + 1\right),$$

where $\rho_i > 0$ and $-1 < \xi_1 < \xi_2 < \cdots < \xi_m \leq 1$.

On the other hand, setting $\theta = 1$ in expansion (11″) we may write the relation

$$\frac{\sigma_0(L)}{x} + \frac{\sigma_1(L)}{x^2} + \ldots + \frac{\sigma_{n+1}(L)}{x^{n+2}} + \frac{1}{x^{n+3}}\,\mathfrak{P}\left(\frac{1}{x}\right) =$$

$$= \frac{1}{x+1}\exp\left\{\frac{1}{2L}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \ldots + \frac{s_n}{x^{n+1}}\right)\right\} =$$

$$= \frac{1}{x+1} + \frac{1}{2L(x+1)}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \ldots + \frac{s_n}{x^{n+1}}\right) + O\left(\frac{1}{L^2}\right).$$

Therefore

$$\frac{\sigma_0(L)}{x} + \frac{\sigma_1(L)}{x^2} + \ldots + \frac{\sigma_{n+1}(L)}{x^{n+2}} =$$

$$= \frac{1}{x+1} + \frac{1}{2L(x+1)}\sum_{i=1}^{m}\frac{\rho_i}{x-\xi_i} + O\left(\frac{1}{L^2}\right) + \frac{1}{x^{n+3}}\,\mathfrak{P}\left(\frac{1}{x}\right) = \frac{1}{2L}\sum_{i=0}^{m}\frac{\mu_i}{x-\xi_i} +$$

$$+ O\left(\frac{1}{L^2}\right) + \frac{1}{x^{n+3}}\,\mathfrak{P}\left(\frac{1}{x}\right),$$

where

$$\xi_0 = -1, \quad \mu_0 = 2L - \sum_{i=1}^{m}\frac{\rho_i}{1+\xi_i}, \quad \mu_i = \frac{\rho_i}{1+\xi_i} \qquad (i = 1; 2, \ldots, m).$$

From this it is apparent that for sufficiently large $L > 0$ the sequence

$$\sigma_0(L), \sigma_1(L), \cdots, \sigma_{n+1}(L)$$

will be positive definite in the interval $< -1, 1 >$ which means the problem (29′), (29″) will have a solution.

Thus we have proved the first part of the following proposition.

Theorem 11. *Let* $\{s_k\}$ *be an arbitrary real sequence if* $-1 < \theta < 1$, *and positive definite (non-negative definite) in the interval* $< -1, 1 >$ *if* $\theta = 1$ *(or* $\theta = -1$*). Then:*

a) *For sufficiently large* $L > 0$ *problem* (29′), (29″) *always has a solution.*

b) *The greatest lower bound* λ *of all such* L *is the largest root of the equation*

$$|\sigma_{i+k}(L)|_0^{m+1} \cdot |\sigma_{i+k}(L) - \sigma_{i+k+2}(L)|_0^m = 0$$

for $n = 2m + 1$ *and of the equation*

$$|\sigma_{i+k}(L) + \sigma_{i+k+1}(L)|_0^m \cdot |\sigma_{i+k}(L) - \sigma_{i+k+1}(L)|_0^m = 0$$

for $n = 2m$, *where the numbers* $\sigma_k(L)$ *are defined by expansion* (11″).

c) *For* $L = \lambda$ *problem* (29′), (29″) *always has as an essentially unique solution* $f_\lambda(u)$ *which is a* λ_θ*-function of order* $p - 1$, *where* $p \leq n + 1$ *is the rank of the sequence*

$$\sigma_0(\lambda), \sigma_1(\lambda), \ldots, \sigma_{n+1}(\lambda).$$

d) *For* $L > \lambda$ *problem* (29′), (29″) *has two and only two solutions which are* L_θ*-functions of order* $n + 1$; *one of these two solutions at the right end of the interval* $< -1, 1>$ *equals* $L(1+\theta)$, *while the other solution does not have this property.*

Proof. Assertion a) is already proved.

Assertion c) follows from Theorem 5 of this chapter.

Assertion b) follows from Theorem 19 of Chapter I.

We turn to assertion d).

Utilizing Theorems 13a, 14a–15a of Chapter I, it is possible to construct exactly two representations of the positive definite sequence $\{\sigma_i(L)\}_0^{n+1}$:

$$\sigma_k(L) = \sum_{j=1}^{m} \rho_j \xi_j^k \qquad (k = 0, 1, 2, \ldots, n+1),$$

where [1]

$$\rho_j > 0, \quad -1 \leq \xi_1 < \xi_2 < \ldots < \xi_m \leq 1,$$

$$\sum_{j=1}^{m} \epsilon_j = n + 2;$$

moreover, in one of these representations $\xi_m = 1$ and in the other $\xi_m < 1$.

The rational function

$$\frac{\psi(x)}{\varphi(x)} = \frac{(x-\eta_1)\ldots(x-\eta_{m-1})}{(x-\xi_1)\ldots(x-\xi_m)} \equiv \sum_{j=1}^{m} \frac{\rho_j}{x-\xi_j} =$$

$$= \frac{\sigma_0}{x} + \frac{\sigma_1(L)}{x^2} + \ldots + \frac{\sigma_{n+1}(L)}{x^{n+2}} + \ldots$$

---

1) Recall that $\epsilon_j = \begin{cases} 2, & \text{if } -1 < \xi_j < 1, \\ 1, & \text{if } |\xi_j| = 1. \end{cases}$

obviously has the property

$$\xi_1 < \eta_1 < \cdots < \eta_{m-1} < \xi_m .$$

Set

$$f_{L_\theta}(u) = L\left\{\theta + \operatorname{sign}\frac{\psi(u)}{\phi(u)}\right\};$$

then, as is easily verified,

$$\int_{-1}^{1} \frac{f_{L_\theta}(u)}{x-u}\,du = 2L\log\left\{\sqrt{x^2-1}\,\left(\frac{x+1}{x-1}\right)^{\frac{\theta}{2}}\frac{\psi(x)}{\varphi(x)}\right\} =$$

$$= 2L\log\left\{\sqrt{x^2-1}\left(\frac{x+1}{x-1}\right)^{\frac{\theta}{2}}\left(\frac{\sigma_0}{x}+\frac{\sigma_1(L)}{x^2}+\cdots+\frac{\sigma_{n+1}(L)}{x^{n+2}}+\cdots\right)\right\}.$$

But then by relation (11″),

$$\int_{-1}^{1} \frac{f_{L_\theta}(u)\,du}{x-u} = \frac{s_0}{x}+\frac{s_1}{x_2}+\cdots+\frac{s_n}{x^{n+1}}+\frac{1}{x^{n+2}}\mathfrak{P}\left(\frac{1}{x}\right).$$

Thus $f_{L_\theta}(u)$ is a solution of problem (29′), (29″). It is also easy to show that $f_{L_\theta}(u)$ has the property discussed in assertion d) and if $\xi_m = 1$, then

$$f_{L_\theta}(u) = L(1+\theta) \quad \text{for} \quad \eta_{m-1} < u \le 1,$$

and if $\xi_m < 1$, then

$$f_{L_\theta}(u) = L(1-\theta) \quad \text{for} \quad \xi_m < u \le 1.$$

Conversely, if $f_{L_\theta}(u)$ is a solution of problem (29′), (29″) which satisfies the conditions of assertion d), then by reversing the reasoning we have gone through we reach the result that the function is generated by one of the two canonical representations of the positive definite sequence $\{\sigma_k(L)\}_0^{n+1}$.

Theorem 11 is therefore proved in its entirety. Assertion d) of Theorem 11 may be generalized to the case when $|\theta| > 1$.

We shall consider in general the exponential problem

$$s_k = \int_{-1}^{1} u^k f(u)\,du \qquad (k = 0, 1, 2, \cdots, n), \tag{*}$$

where we place upon $f(u)$ the restriction

$$\alpha \le f(u) \le \beta. \tag{**}$$

We shall say that a function $f_0(u)$ is an (α, β)-function of order m if it is equal to α or β everywhere and has exactly m points of discontinuity, all of which lie inside the interval $< -1, 1>$. Now let the sequence $\{s_k\}_0^n$ have the property that problem (*), (**) has no solution which is an (α, β)-function of order $< n$. Then we assert that problem (*), (**) has exactly two solutions, which are (α, β)-functions of order n, one equal to β and the other equal to α at the point $+1$. It is easiest to verify this if we replace $f(u)$ by the function $f(u) - (\alpha + \beta)/2$ in our problem and in the corresponding manner transform the sequence of moments $\{s_k\}_0^n$.

The analogous proposition for the trigonometric L-problem is valid.[1)]

In order not to repeat our derivations we shall restrict our formulation to the simplest case:

Theorem 11a. *Let*

$$c_0 \gtrless 0,\ c_1,\ c_2,\ \cdots,\ c_{n-1}$$

be an arbitrary complex sequence.

a) *For sufficiently large $L > 0$ the problem*

$$\begin{aligned} -L \le f(t) \le L \qquad & (0 \le t \le 2\pi) \\ c_k = \int_0^{2\pi} f(t)\, e^{ikt} dt \qquad & (k = 0,\ 1,\ 2,\ \cdots,\ n-1) \end{aligned} \tag{2}$$

always has a solution.

b) *The greatest lower bound λ of all such L is the largest root of the equation*

$$\begin{vmatrix} \gamma_0(L) & \gamma_1(L) & \cdots & \gamma_{n-1}(L) \\ \gamma_{-1}(L) & \gamma_0(L) & \cdots & \gamma_{n-2}(L) \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{-(n-1)}(L) & \gamma_{-(n-2)}(L) & \cdots & \gamma_0(L) \end{vmatrix} = 0,$$

where the numbers $\gamma_k(L) = \gamma_k$ are determined from the expansion

$$\exp\left\{\frac{i}{2L}\left(\frac{c_0}{2} + c_1 z + \cdots + c_{n-1} z^{n-1}\right)\right\} = \gamma + \gamma_1 z + \cdots + \gamma_{n-1} z^{n-1} + \cdots,$$

$$\gamma_0 = \gamma + \bar{\gamma} = 2\cos\frac{c_0}{4L},\quad \gamma_{-k} = \bar{\gamma}_k. \tag{4}$$

c) *For $L = \lambda$ problem (2) always has an essentially unique solution $f_\lambda(t)$*

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1) Cf. N. Ahiezer and M. Kreĭn [1a, 1i].

which is a λ*-function of order* 2p, *where* p *is the rank of the sequence*

$$\gamma_0(\lambda),\ \gamma_1(\lambda),\ \cdots,\ \gamma_{n-1}(\lambda).$$

We note that the construction of the function $f_\lambda(t)$ from the coefficients $c_0,\ c_1,\ \cdots,\ c_{n-1}$ may be completed by the means set forth in the proof of Theorem 1.

Example 1. Let there be given

$$c_0 = c_1 = \cdots = c_{n-3} = 0,\ c_{n-2},\ c_{n-1} \qquad (n > 3).$$

Equation (4) takes the form

$$\exp\left\{\frac{i}{2L}\left(c_{n-2}z^{n-2} + c_{n-1}z^{n-1}\right)\right\} = \gamma + \gamma_1 z + \cdots + \gamma_{n-1}z^{n-1} + \cdots$$

which gives

$$\gamma = 1,\ \gamma_1 = \cdots = \gamma_{n-3} = 0,$$

$$\gamma_{n-2} = \frac{i}{2L}\,c_{n-2},\ \gamma_{n-1} = \frac{i}{2L}\,c_{n-1}.$$

Thus the equation for finding λ is

$$\begin{vmatrix} 2 & 0 & \cdots & 0 & \frac{i}{2L}c_{n-2} & \frac{i}{2L}c_{n-1} \\ 0 & 2 & \cdots & 0 & 0 & \frac{i}{2L}c_{n-2} \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{i}{2L}c_{n-2} & 0 & \cdots & 0 & 2 & 0 \\ -\frac{i}{2L}\bar{c}_{n-1} & -\frac{i}{2L}\bar{c}_{n-2} & \cdots & 0 & 0 & 2 \end{vmatrix} = 0$$

or

$$16L^4 - L^2(2|c_{n-2}|^2 + |c_{n-1}|^2) + \frac{1}{16}|c_{n-2}|^4 = 0.$$

Hence

$$\lambda = \frac{1}{8}\left\{|c_{n-1}| + \sqrt{|c_{n-1}|^2 + 4|c_{n-2}|^2}\right\}.$$

Thus, if

$$f(t) \sim \frac{1}{2\pi}\cdot\sum_{|k|\geq n-2} c_k e^{-ikt} \qquad (c_k = \bar{c}_k),$$

then

$$\operatorname{vrai\,max}_{0\leq t\leq 2\pi}|f(t)| \geq \frac{1}{8}\left\{|c_{n-1}| + \sqrt{|c_{n-1}|^2 + 4|c_{n-2}|^2}\right\}.$$

Example 2. Let there be given

$$c_0 > 0,\ c_1 = \cdots = c_{n-2} = 0,\ c_{n-1} \qquad (n > 2).$$

Here

$$\gamma_0 = 2\cos\frac{c_0}{4L},\ \gamma_1 = \cdots = \gamma_{n-2} = 0,\ \gamma_{n-1} = \frac{i}{2L}\, c_{n-1}\cdot e^{\frac{ic_0}{4L}}.$$

Therefore the equation for determining λ takes the form

$$\cos^{n-2}\frac{\pi c_0}{4L}\left\{\cos^2\frac{c_0}{4L} - \frac{1}{16L^2}\,|c_{n-1}|^2\right\} = 0.$$

The largest root of the equation

$$\cos\frac{\pi c_0}{4L} = 0$$

is $\frac{1}{2}c_0$, whereas the equation

$$\cos^2\frac{c_0}{4L} - \frac{1}{16L^2}\,|c_{n-1}|^2 = 0$$

has for $c_{n-1} \neq 0$ a larger root, since its left side is negative for $L = \frac{1}{2}\,c_0$ and positive for $L = \infty$.

Thus, if

$$f(t) \sim \frac{c_0}{2\pi} + \frac{1}{2\pi}\sum_{|k|\geq n-1} c_k\, e^{-ikt} \qquad (c_0 > 0,\ n > 2),$$

then

$$\operatorname{vrai}\max_{0\leq t\leq 2\pi}|f(t)| \geq \lambda,$$

where λ is the largest root of the equation

$$\cos\frac{c_0}{4L} = \frac{1}{4L}\,|c_{n-1}|.$$

We now turn our attention to several minimum-problems which are related to Theorems 11 and 11a.

Problem 1. Given the real numbers

$$s_0,\ s_1,\ \cdots,\ s_n \qquad \left(\sum_{i=0}^{n} s_i^2 > 0\right)$$

and some number θ $(-1 < \theta < 1)$. Consider the set of all polynomials

$$P(u) = a_0 + a_1 u + \cdots + a_n u^n,$$

whose coefficients $a_0,\ a_1,\ \cdots,\ a_n$ satisfy the relation

$$a_0 s_0 + a_1 s_1 + \cdots + a_n s_n = 1. \tag{30}$$

It is required to find the minimum of the integral

$$I[P] = \int_{-1}^{1} \{ |P(u)| + \theta P(u) \}\, du,$$

and also all those polynomials $P(u)$ for which this minimum is attained.

Remark. We obtain the Korkin–Zolotarev problem (which in essence, as was very kindly pointed out to us by Ja. L. Geronymus, was already solved by P. Čebyšev [18a] and then solved again by Korkin and Zolotarev [7], Stieltjes [39b], Fujiwara [26a] and S. N. Bernštein [3a]), if we set $\theta = 0$, $s_0 = s_1 = \cdots = s_{n-1} = 0$, $s_n = 1$.

Solution. By assertion c) of Theorem 11 the solution of the problem

$$s_k = \int_{-1}^{1} u^k f(u)\, du \quad (k = 0, 1, \cdots, n)$$
$$-L(1-\theta) \le f(u) \le L(1+\theta)$$

for the smallest value L, i.e., for $L = \lambda$, has the form

$$f_\lambda(u) = \lambda \{\theta + \operatorname{sign} \Phi(u)\},$$

where $\Phi(u)$ is a polynomial of degree $< n$.

Further, let $P(u)$ be an arbitrary polynomial of degree $\le n$ satisfying relation (30). Then

$$1 = \sum_{i=0}^{n} s_i a_i = \sum_{i=0}^{n} a_i \int_{-1}^{1} u^i f_\lambda(u)\, du = \int_{-1}^{1} P(u) f_\lambda(u)\, du =$$

$$= \lambda \int_{-1}^{1} P(u) \{\theta + \operatorname{sign} \Phi(u)\}\, du \le \lambda \int_{-1}^{1} \{ |P(u)| + \theta P(u) \}\, du$$

and this means

$$I[P] \ge \frac{1}{\lambda}.$$

The equality sign in this relation is obviously taken on only when $P(u)$ is a polynomial (which satisfies relation (30) and is of degree $\le n$) for which

$$|P(u)| = P(u) \operatorname{sign} \Phi(u).$$

We shall show that such a polynomial does indeed exist.

To this end take an arbitrary polynomial $\Omega(u) \not\equiv 0$ of degree $n - p + 1$ which

is non-negative in the interval $-1 < u < 1$.

Since

$$f_\lambda(u)\,\Phi(u) \geq 0 \qquad (-1 < u < 1),$$

therefore

$$\int_{-1}^{1} f_\lambda(u)\,\Phi(u)\,\Omega(u)\,du = C > 0.$$

We set

$$P_0(u) = \frac{\Phi(u)\,\Omega(u)}{C},$$

so that

$$\int_{-1}^{1} P_0(u)\,f_\lambda(u)\,du = 1,$$

which means the polynomial $P_0(u)$ of degree n satisfies condition (30).

Moreover,

$$|P_0(u)| = \frac{|\Phi(u)|\,\Omega(u)}{C} = \frac{\Phi(u)\,\Omega(u)}{C}\operatorname{sign}\Phi(u) = P_0(u)\operatorname{sign}\Phi(u),$$

from which by the above remark

$$I[P_0] = \frac{1}{\lambda}.$$

Thus *the minimum of the integral* $I[P]$ *is* $\frac{1}{\lambda}$, *where* λ *is the number determined in Theorem* 11, *and this minimum is attained if and only if*

$$P(u) \equiv P_0(u) = \frac{\Phi(u)\,\Omega(u)}{C},$$

where $\Omega(u) \not\equiv 0$ *is an arbitrary polynomial of degree* $\leq n - p + 1$ *which is non-negative in the interval* $-1 < u < 1$ *and*

$$C = \int_{-1}^{1} f_\lambda(u)\,\Phi(u)\,\Omega(u)\,du;$$

the polynomial $P_0(u)$ *is unique, that is* $\Omega(u)$ *is a constant if and only if* $p = n+1$, *i.e., if of the two forms*

$$\sum_{i,k=0}^{m+1} \sigma_{i+k}(\lambda)\,x_i x_k, \qquad \sum_{i,k=0}^{m} [\sigma_{i+k}(\lambda) - \sigma_{i+k+2}(\lambda)]\,x_i\,x_k$$

for $n = 2m + 1$ *and of the two forms*

$$\sum_{i,k=0}^{m} [\sigma_{i+k}(\lambda) + \sigma_{i+k+1}(\lambda)]\,x_i\,x_k, \qquad \sum_{i,k=0}^{m} [\sigma_{i+k}(\lambda) - \sigma_{i+k+1}(\lambda)]\,x_i\,x_k$$

for $n = 2m$ one is positive.

Example. Let $s_0 = s_1 = \cdots = s_{n-1} = 0$, $s_n = 1$. The expansion (11″) then has the form

$$\frac{1}{\sqrt{x^2-1}}\left(\frac{x-1}{x+1}\right)^{\frac{\theta}{2}} e^{\frac{1}{2Lx^{n+1}}} = \frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \cdots + \frac{\sigma_n}{x^{n+1}} + \left(\sigma_{n+1} + \frac{1}{2L}\right)\frac{1}{x^{n+2}} + \cdots,$$

where $\sigma_0, \sigma_1, \cdots, \sigma_{n+1}, \cdots$ are determined by the series

$$\frac{1}{\sqrt{x^2-1}}\left(\frac{x-1}{x+1}\right)^{\frac{\theta}{2}} = \frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \cdots + \frac{\sigma_n}{x^{n+1}} + \frac{\sigma_{n+1}}{x^{n+2}} + \cdots \qquad (\sigma_0 = 1).$$

We already observed above that

$$\frac{1}{\sqrt{x^2-1}}\left(\frac{x-1}{x+1}\right)^{\frac{\theta}{2}} = \frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\left(\frac{1-u}{1+u}\right)^{\frac{\theta}{2}}\frac{du}{x-u}.$$

Therefore

$$\frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\left(\frac{1-u}{1+u}\right)^{\frac{\theta}{2}}\frac{du}{x-u} = \frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \cdots + \frac{\sigma_{n+1}}{x^{n+2}} + \cdots, \qquad (\alpha)$$

and we obtain analogously that

$$\frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{1}\left(\frac{1-u}{1+u}\right)^{\frac{1+\theta}{2}}\frac{du}{x-u} = \frac{\sigma_0-\sigma_1}{x} + \frac{\sigma_1-\sigma_2}{x^2} + \cdots + \frac{\sigma_n-\sigma_{n+1}}{x^{n+1}} + \cdots \qquad (\beta)$$

$$\frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{1}\left(\frac{1-u}{1+u}\right)^{\frac{\theta-1}{2}}\frac{du}{x-u} = \frac{\sigma_0+\sigma_1}{x} + \frac{\sigma_1+\sigma_2}{x^2} + \cdots + \frac{\sigma_n+\sigma_{n+1}}{x^{n+1}} + \cdots \qquad (\gamma)$$

$$\frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{1}\sqrt{1-u^2}\left(\frac{1-u}{1+u}\right)^{\frac{\theta}{2}}\frac{du}{x-u} = \frac{\sigma_0-\sigma_2}{x} + \frac{\sigma_1-\sigma_3}{x^2} + \cdots + \frac{\sigma_{n-1}-\sigma_{n+1}}{x^n} + \cdots. \qquad (\delta)$$

From this it follows that the determinants

$$\left|\sigma_{i+k}\right|_0^m,\ \left|\sigma_{i+k}+\sigma_{i+k+1}\right|_0^m,\ \left|\sigma_{i+k}-\sigma_{i+k+1}\right|_0^m,\ \left|\sigma_{i+k}-\sigma_{i+k+2}\right|_0^m$$

are positive.

Therefore λ is a root of the equation

$$\begin{vmatrix} \sigma_0 - \sigma_1 & \sigma_1 - \sigma_2 & \cdots \sigma_m - \sigma_{m+1} \\ \sigma_1 - \sigma_2 & \sigma_2 - \sigma_3 & \cdots \sigma_{m+1} - \sigma_{m+2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_m - \sigma_{m+1} & \sigma_{m+1} - \sigma_{m+2} & \cdots \sigma_{2m} - \sigma_{2m+1} - \frac{1}{2L} \end{vmatrix} = 0 \qquad (\epsilon)$$

for $n = 2m$ and of the equation

$$\begin{vmatrix} \sigma_0 - \sigma_2 & \sigma_1 - \sigma_3 & \cdots \sigma_m - \sigma_{m+2} \\ \sigma_1 - \sigma_3 & \sigma_2 - \sigma_4 & \cdots \sigma_{m+1} - \sigma_{m+3} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_m - \sigma_{m+2} & \sigma_{m+1} - \sigma_{m+3} & \cdots \sigma_{2m} - \sigma_{2m+2} - \frac{1}{2L} \end{vmatrix} = 0 \qquad (\zeta)$$

for $n = 2m + 1$.

Thus

$$\lambda = \frac{1}{2} \left| \frac{\sigma_{i+k} - \sigma_{i+k+1} \Big|_0^{m-1}}{\sigma_{i+k} - \sigma_{i+k+1} \Big|_0^{m}} \right| \qquad (\epsilon')$$

for $n = 2m$ and

$$\lambda = \frac{1}{2} \left| \frac{\sigma_{i+k} - \sigma_{i+k+2} \Big|_0^{m-1}}{\sigma_{i+k} - \sigma_{i+k+2} \Big|_0^{m}} \right| \qquad (\zeta')$$

for $n = 2m + 1$.

It is not difficult to calculate λ.

Indeed, on the basis of expansions (β) and (δ) and Section 2 of §1 of Chapter I, λ is half the square of the first coefficient of the polynomial of degree m orthonormalized relative to the weight

$$p_1(u) = \frac{1}{\pi} \cos \frac{\pi\theta}{2} \left(\frac{1-u}{1+u}\right)^{\frac{1+\theta}{2}}$$

for $n = 2m$ and relative to the weight

$$p_2(u) = \frac{1}{\pi} \cos \frac{\pi\theta}{2} \sqrt{1-u^2} \left(\frac{1-u}{1+u}\right)^{\frac{\theta}{2}}$$

for $n = 2m + 1$, since

$$\sigma_k - \sigma_{k+1} = \int_{-1}^{1} p_1(u)\, u^k du, \quad \sigma_k - \sigma_{k+2} = \int_{-1}^{1} p_2(u)\, u^k du \qquad (k = 0, 1, 2, \cdots).$$

Since we have here the weight to which the Jacobi polynomials correspond,

thus on the basis of known expressions for Jacobi polynomials (cf. for example V. Gončarov [5], pp. 91–94) we have:

$$\lambda = \frac{1}{4^{m+1}} \frac{\binom{2m}{m}\binom{2m+1}{m}}{\binom{m-\frac{1+\theta}{2}}{m}\binom{m+\frac{1+\theta}{2}}{m+1}} \qquad (\epsilon'')$$

for $n = 2m$ and

$$\lambda = \frac{1}{4^{m+1}} \frac{\binom{2m+1}{m}\binom{2m+1}{m}}{\binom{m+\frac{1+\theta}{2}}{m+1}\binom{m+\frac{1-\theta}{2}}{m+1}} \qquad (\zeta'')$$

for $n = 2m + 1$.

It remains for us to find the extremal polynomial $P_0(u)$, in other words the polynomial $\Phi(u)$ whose roots lie in the interval $-1 < u < 1$ and through which the function $f_\lambda(u)$ of Theorem 11 is represented in the form

$$f_\lambda(u) = \lambda\{\theta + \operatorname{sign}\Phi(u)\}.$$

By the remark on page 100 the extremal polynomial $P_0(u)$ is unique, which means $\Phi(u)$ is a polynomial of exactly nth degree.

Consider the case when $n = 2m$.

Denoting the roots of the polynomial $P_0(u)$ by

$$\xi_1 < \eta_1 < \xi_2 < \cdots < \xi_m < \eta_m \qquad (\xi_1 > -1,\ \eta_m < 1),$$

we have

$$P_0(u) \equiv \epsilon\Phi(u) = \phi(u)\psi(u) \qquad (\epsilon = \pm 1),$$

where

$$\phi(u) = (u - \xi_1) \cdots (u - \xi_m),$$

$$\psi(u) = (u - \eta_1) \cdots (u - \eta_m).$$

Thus,

$$\frac{1}{\sqrt{x^2-1}} \left(\frac{x-1}{x+1}\right)^{\frac{\theta}{2}} \exp\left\{\frac{1}{2}\int_{-1}^{1} \frac{f_\lambda(u)\,du}{x-u}\right\} = \frac{1}{\sqrt{x^2-1}} \exp\left\{\frac{1}{2}\int_{-1}^{1} \frac{\operatorname{sign}\Phi(u)}{x-u}\,du\right\} =$$

$$= \frac{1}{\sqrt{x^2-1}} \left(\frac{x+1}{x-1}\right)^{\frac{\epsilon}{2}} \left[\frac{\psi(x)}{\phi(x)}\right]^{\epsilon}. \qquad (\eta)$$

But the expression obtained coincides with the series

$$\frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \cdots + \frac{\sigma_{2m}}{x^{2m+1}} + (\sigma_{2m+1} + \frac{1}{2\lambda}) \frac{1}{x^{2m+2}} + \cdots, \qquad (\eta')$$

where λ is a root of the equation (ϵ).

Therefore the coefficients of the series (η') may be written as

$$\sigma_k = \sum_{i=1}^{m+1} \rho_i \alpha_i^k \qquad (k = 0, 1, \cdots, 2m)$$

$$\sigma_{2m+1} + \frac{1}{2\lambda} = \sum_{i=1}^{m+1} \rho_i \alpha_i^{2m+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

where $\rho_i > 0$ and

$$-1 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} = 1.$$

From this it follows that the function (η) equals

$$\sum_{i=1}^{m+1} \frac{\rho_i}{x - \alpha_i}.$$

This means $\epsilon = 1$ and $\alpha_i = \xi_i$ $(i = 1, 2, \cdots, m)$.

However we see that

$$\frac{1}{x-1} \frac{\psi(x)}{\phi(x)} = \frac{\sigma_0}{x} + \frac{\sigma_1}{x^2} + \cdots + \frac{\sigma_{2m}}{x^{2m+1}} + (\sigma_{2m+1} + \frac{1}{2\lambda}) \frac{1}{x^{2m+2}} + \cdots.$$

Thus

$$\frac{\psi(x)}{\phi(x)} = 1 + \frac{\sigma_1 - \sigma_0}{x} + \frac{\sigma_2 - \sigma_1}{x^2} + \cdots + \frac{\sigma_{2m} - \sigma_{2m-1}}{x^{2m}} + \cdots$$

and in view of (β)

$$\frac{\phi(x) - \psi(x)}{\phi(x)} + \frac{1}{x^{2m+1}} \mathfrak{P}(\frac{1}{x}) = \frac{1}{\pi} \cos \frac{\pi\theta}{2} \int_{-1}^{1} (\frac{1-u}{1+u})^{\frac{1+\theta}{2}} \frac{du}{x-u} = 1 - (\frac{x-1}{x+1})^{\frac{1+\theta}{2}}.$$

Hence

$$\frac{\psi(x)}{\phi(x)} + \frac{1}{x^{2m+1}} \mathfrak{P}(\frac{1}{x}) = (\frac{x-1}{x+1})^{\frac{1+\theta}{2}}$$

which means

$$\frac{\phi(x)}{\psi(x)} + \frac{1}{x^{2m+1}} \mathfrak{P}_1(\frac{1}{x}) = (\frac{x+1}{x-1})^{\frac{1+\theta}{2}},$$

$$\frac{\phi(x)-\psi(x)}{\psi(x)}+\frac{1}{x^{2m+1}}\mathfrak{P}_1\left(\frac{1}{x}\right)=\left(\frac{x+1}{x-1}\right)^{\frac{1+\theta}{2}}-1=\frac{1}{\pi}\cos\frac{\pi\theta}{2}\int_{-1}^{1}\left(\frac{1+u}{1-u}\right)^{\frac{1+\theta}{2}}\frac{du}{x-u}.$$

The formulas found indicate that $\phi(u)$ and $\psi(u)$ are orthogonal polynomials of degree m with unity as the coefficients of u^m relative to the weights

$$\frac{1}{\pi}\cos\frac{\pi\theta}{2}\left(\frac{1-u}{1+u}\right)^{\frac{1+\theta}{2}},\quad \frac{1}{\pi}\cos\frac{\pi\theta}{2}\left(\frac{1+u}{1-u}\right)^{\frac{1+\theta}{2}}.$$

Thus denoting as is usual by $J_n^{(\alpha,\beta)}(u)$ the Jacobi polynomial of degree n relative to the weight $(1-u)^\alpha(1+u)^\beta$ it is not difficult to obtain the extremal polynomial $P_0(u)$ in the form

$$4^m\left\{\frac{m!\,m!}{(2m)!}\right\}^2 J_m^{\left(\frac{1+\theta}{2},-\frac{1+\theta}{2}\right)}(u)\,J_m^{\left(-\frac{1+\theta}{2},\frac{1+\theta}{2}\right)}(u).$$

Analogously, for $n = 2m + 1$ we find that

$$P_0(u)=2\cdot 4^m\left\{\frac{m!\,(m+1)!}{(2m+1)!}\right\}^2 J_m^{\left(\frac{1+\theta}{2},\frac{1-\theta}{2}\right)}(u)\,J_m^{\left(-\frac{1+\theta}{2},-\frac{1-\theta}{2}\right)}(u).$$

Problem 2. Given a complex sequence

$$c_0 \gtreqless 0, c_1, c_2, \cdots, c_{n-1}.$$

Consider the set of all trigonometric sums

$$T(\phi)=\sum_{k=-(n-1)}^{n-1}\xi_k e^{ik\phi},$$

for which

$$\int_0^{2\pi}|T(\phi)|\,d\phi\le 1.$$

In addition let

$$T_c(\phi)=\sum_{k=-(n-1)}^{n-1}c_k\xi_k e^{ik\phi}\qquad(c_{-k}=\bar{c}_k);$$

it is required to find

$$\sup\max_{0\le\phi\le 2\pi}|T_c(\phi)|.$$

Solution. Applying Theorem 11a to the sequence

$$c_0, c_1, \cdots, c_{n-1}$$

we find the λ-function $f_\lambda(t)$ of order $< 2n$ for which

$$c_k = \int_0^{2\pi} f_\lambda(t) e^{ikt} dt \qquad (k = 0, 1, \cdots, n-1).$$

It is therefore easy to see that

$$T_c(\phi) = \int_0^{2\pi} f_\lambda(\psi) T(\phi + \psi) d\psi.$$

Hence

$$|T_c(\phi)| \leq \int_0^{2\pi} |f_\lambda(\psi)| \cdot |T(\phi + \psi)| d\psi = \lambda \int_0^{2\pi} |T(\phi + \psi)| d\psi \leq \lambda.$$

On the other hand, taking for $T(\phi)$ the trigonometric sum $T^*(\phi)$ of order $\leq n-1$, which changes sign at those points (and there are $\leq 2n-2$) where $f_\lambda(\phi)$ changes sign and which has been normalized in accordance with the equality

$$\int_0^{2\pi} |T^*(\phi)| d\phi = 1,$$

we find that

$$T_c^*(0) = \int_0^{2\pi} f_\lambda(\psi) T^*(\psi) d\psi = \int_0^{2\pi} |f_\lambda(\psi)| \cdot |T^*(\psi)| d\psi = \lambda.$$

Thus, the value sought equals the number λ determined in Theorem 11a.

Thus, whatever the sequence

$$c_0 \gtrless 0, c_1, \cdots, c_{n-1},$$

an equation may be displayed whose largest root is [1]

---

1) In particular, taking $c_k = 1$ $(k = 0, 1, \cdots, n-1)$, we obtain a (transcendental) equation for the quantity

$$\sup_{\xi_k} \frac{\max\limits_{0 \leq \varphi \leq 2\pi} |T(\varphi)|}{\int_0^{2\pi} |T(\varphi)| d\varphi},$$

and taking $c_k = ik$ $(k = 0, 1, \cdots, n-1)$, we find an algebraic equation for the quantity

$$\sup_{\xi_k} \frac{\max\limits_{0 \leq \varphi \leq 2\pi} |T'(\varphi)|}{\int_0^{2\pi} |T(\varphi)| d\varphi}.$$

$$\lambda = \sup_{\xi_k} \frac{\max\limits_{0\leqslant\varphi\leqslant 2\pi} \left| \sum\limits_{-(n-1)}^{n-1} c_k \xi_k e^{ik\varphi} \right|}{\int\limits_0^{2\pi} \left| \sum\limits_{-(n-1)}^{n-1} \xi_k e^{ik\varphi} \right| d\varphi}.$$

It is easy to see that the same value λ gives the quantity

$$\sup_{\xi_k} \frac{\left| \sum\limits_{-(n-1)}^{n-1} c_k \xi_k \right|}{\int\limits_0^{2\pi} \left| \sum\limits_{-(n-1)}^{n-1} \xi_k e^{ik\varphi} \right| d\varphi};$$

in other words, $\frac{1}{\lambda}$ is

$$\min_{\xi_k} \int\limits_0^{2\pi} \left| \sum_{-(n-1)}^{n-1}{}' \xi_k e^{ik\phi} \right| d\phi$$

under the condition

$$\sum_{-(n-1)}^{n-1} c_k \xi_k = 1. \qquad (*)$$

The circumstance that the reciprocal of the minimum of the integral

$$\int\limits_0^{2\pi} \left| \sum_{-(n-1)}^{n-1} \xi_k e^{ik\phi} \right| d\phi \qquad (**)$$

under the condition (*) is the same as the greatest lower bound for those L for which problem (2) has a solution, was called to our attention by Ja. L. Geronymus, who considered real trigonometric sums ($\xi_{-k} = \overline{\xi_k}$) and proposed for the proof somewhat different reasoning. [1]

We note also that recently Ja. L. Geronymus succeeded in finding a solution to the problem of minimizing the integral (**) (for $\xi_{-k} = \overline{\xi_k}$) if we are given s of the higher coefficients ξ_k, where $s < \left[\frac{2n}{3}\right]$.

§5

1. To conclude the present chapter we consider one application of the results

1) This fact, as in the analogous case of the exponential problem, is in essence a simple consequence of certain general results already indicated by F. Riesz and Hahn, for which the reader should see §3 of Article IV of this book.

we have found to the question of construction of a special form of the quadrature formulas which first was done by P. Čebyšev [18b], and was later considered by A. A. Markov under somewhat different conditions.

P. Čebyšev in §7 of his memoir states the problem: *find approximately the integral*

$$\int_{-1}^{1} f(x)\,q(x)\,dx,$$

where the given function $q(x)$ *satisfies the condition*

$$\int_{-1}^{1} q(x)\,dx = 0 \tag{31}$$

with the help of the formula

$$\int_{-1}^{1} f(x)\,q(x)\,dx = K \sum_{i=1}^{m} \{f(\alpha_i) - f(\beta_i)\}. \tag{32}$$

Because the right-hand side of equality (32) contains $2m + 1$ parameters (the real number K and the nodes α_i, β_i) which lie in the interval $-1 \leq x \leq 1$, to find it P. Čebyšev requires that formula (32) be exact if $f(x)$ is an arbitrary polynomial of degree $\leq 2m + 1$ (the free term of the polynomial plays no role by condition (31)).

In as much as

$$\frac{1}{z-x} = P_{2m+1}(x) + \frac{x^{2m+2}}{z^{2m+2}(z-x)},$$

where $P_{2m+1}(x)$ is a polynomial of degree $2m + 1$ in x, on the basis of the requirements which have been shown for the existence of a quadrature formula (32) the equality

$$\int_{-1}^{1} \frac{q(x)\,dx}{z-x} = K \sum_{i=1}^{m} \left\{ \frac{1}{z-\alpha_i} - \frac{1}{z-\beta_i} \right\} + \frac{1}{z^{2m+3}}\,\mathfrak{P}\left(\frac{1}{z}\right)$$

must hold from which

$$\int_{-1}^{1} q(x) \log(z-x)\,dx = K \log \frac{\phi(z)}{\psi(z)} + \frac{1}{z^{2m+2}}\,\mathfrak{P}_1\left(\frac{1}{z}\right),$$

where

$$\phi(z) = (z-\alpha_1)\cdots(z-\alpha_m), \quad \psi(z) = (z-\beta_1)\cdots(z-\beta_m),$$

and thus

$$e^{-\frac{1}{K}\int_{-1}^{1} q(x)\log(z-x)\,dx} = \frac{\psi(z)}{\phi(z)} + \frac{1}{z^{2m+2}}\,\mathfrak{P}_2\left(\frac{1}{z}\right). \tag{33}$$

By condition (31) we may write

$$\int_{-1}^{1} q(x)\log(z-x)\,dx = \int_{-1}^{1} q(x)\log\left(1-\frac{x}{z}\right)dx = -\sum_{\nu=1}^{\infty}\frac{1}{\nu z^{\nu}}\int_{-1}^{1} q(x)x^{\nu}\,dx.$$

Therefore, introducing the notation

$$s_k = \frac{1}{k+1}\int_{-1}^{1} q(x)x^{k+1}\,dx \qquad (k = 0, 1, \cdots, 2m), \tag{34}$$

we obtain by (33) the relation

$$e^{\frac{1}{K}\left\{\frac{s_0}{z}+\frac{s_1}{z^2}+\cdots+\frac{s_{2m}}{z^{2m+1}}\right\}} = \frac{\psi(z)}{\phi(z)} + \frac{1}{z^{2m+2}}\mathfrak{P}_3\left(\frac{1}{z}\right). \tag{33'}$$

Therefore on the basis of definition (13) of the quantities $t_k(L)$:

$$1 + \frac{t_0(K)}{z} + \frac{t_1(K)}{z^2} + \cdots + \frac{t_{2m}(K)}{z^{2m+1}} + \cdots = \frac{\psi(z)}{\phi(z)}. \tag{35}$$

Thus, *if the quadrature formula* (32) *holds there must exist polynomials* $\phi(z)$ *and* $\psi(z)$ *of degree* m *with roots in the interval* $<-1, 1>$ *and a real number* K *for which relation* (35) *is fulfilled.*

This result shows that the number K must satisfy the equation

$$\begin{vmatrix} t_0(L) & \cdots & t_m(L) \\ \cdots & \cdots & \cdots \\ t_m(L) & \cdots & t_{2m}(L) \end{vmatrix} \equiv \Delta_m(L) = 0 \tag{36}$$

and this gives us the following rule for construction of the quadrature formula (32): *upon finding a root* K *of equation* (36) *it is necessary to define the fraction* $\frac{\psi(z)}{\phi(z)}$ *of degree* m *by using the expansion* (35), *after which the roots* α_i, β_i *are determined as roots of the denominator and numerator of the fraction in question.*

P. Čebyšev himself gave the rule in a somewhat different form, using an algorithm of continued fractions; we shall not stop for his formulation, but we remark that both the rule we have deduced and P. Čebyšev's rule immediately raise the following questions:

1) does equation (36) have a real root?

2) if there is a real root, does there exist a fraction $\frac{\psi(z)}{\phi(z)}$ which satisfies relation (35)?

3) if there is such a fraction, will the roots of its numerator and denominator lie in the interval $<-1, 1>$?

P. Čebyšev considered none of these questions, although it is possible to give simple examples for which these questions must be answered negatively. We shall restrict ourselves to a single example.

Let

$$q(x) = Ax + Bx^3 + Cx^5,$$

where A, B and C are determined from the condition $s_0 = 1$, $s_1 = s_2 = s_3 = 0$, $s_4 = -1$; a simple calculation gives

$$\Delta_2(L) = \frac{\text{const}}{L^5}\left(\frac{1}{L^4} - 720\right),$$

$$\frac{\psi(z)}{\phi(z)} = \frac{12K^2z^2 + 6Kz + 1}{12K^2z^2 - 6Kz + 1},$$

from which

$$\alpha_1 = \frac{1}{4K}\left(1 + \frac{i}{\sqrt{3}}\right),$$

$$\alpha_2 = \frac{1}{4K}\left(1 - \frac{i}{\sqrt{3}}\right).$$

Twenty years after P. Čebyšev's memoir, there appeared a memoir by A. A. Markov [13b] in which A. A. Markov treated the question of construction of the quadrature formulas (32).

A. A. Markov, weakening P. Čebyšev's condition slightly, imposed the requirement that formula (32) be correct for any polynomial $f(x)$ of degree $\leq 2m$.

As a result, A. A. Markov was able to establish a very large class of functions $q(x)$ for which formula (32) was valid; and more than that, for any m A. A. Markov found an infinite number of formulas (32) and just one formula for every value K from some interval.

Characteristic of the formulas of A. A. Markov is the alternation of the nodes α_i with the nodes β_i.

A. A. Markov did not, however, find necessary and sufficient conditions for his formulas to exist. At the same time only by solving this question is one able[1)]

1) Cf. N. Ahiezer and M. Kreĭn [1h].

to discover the basic connection between the formulas of A. A. Markov and one type of the formulas of P. Čebyšev, and by the same token to investigate the latter.

2. We shall first consider the case of the infinite interval.

For a real function $q(x)$ $(-\infty < x < \infty)$ let the integral

$$\int_{-\infty}^{\infty} x^k q(x)\,dx = k s_{k-1}$$

make sense for all positive integers k which we shall consider and let

$$\int_{-\infty}^{\infty} q(x)\,dx = 0.$$

We shall say the function $q(x)$ admits a *quadrature formula of A. A. Markov* of order $2m$ if there exist numbers

$$\eta_1 \le \xi_1 \le \eta_2 \le \cdots \le \eta_m \le \xi_m \tag{37}$$

and a real number L for which

$$\int_{-\infty}^{\infty} P_{2m}(x) q(x)\,dx = L \sum_{i=1}^{m} \{P_{2m}(\xi_i) - P_{2m}(\eta_i)\}, \tag{38}$$

whatever the polynomial $P_{2m}(x)$ of degree $\le 2m$.

Furthermore, we shall say $q(x)$ admits a *quadrature formula of P. Čebyšev* of order $2m + 1$ if there exist real numbers K, α_i and β_i for which

$$\int_{-\infty}^{\infty} P_{2m+1}(x) q(x)\,dx = K \sum_{i=1}^{m} \{P_{2m+1}(\alpha_i) - P_{2m+1}(\beta_i)\}, \tag{39}$$

whatever the polynomial $P_{2m+1}(x)$ of degree $\le 2m + 1$.

Theorem 12. *In order that the function $q(x)$ admit a quadrature formula of A. Markov of order $2m$ it is necessary and sufficient that the sequence*

$$s_0, s_1, \cdots, s_{2m-2}$$

be positive definite or negative definite in the interval $(-\infty, \infty)$.

Proof. Assume the function $q(x)$ admits the quadrature formula of A. A. Markov (38) of order $2m$.

From this formula it follows that

$$(k+1) s_k = \int_{-\infty}^{\infty} x^{k+1} q(x)\,dx = L \sum_{i=1}^{m} \{\xi_i^{k+1} - \eta_i^{k+1}\} \qquad (k = 0, 1, \cdots, 2m-1). \tag{40}$$

Setting

$$\phi(x) = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_m),$$

$$\psi(x) = (x - \eta_1)(x - \eta_2) \cdots (x - \eta_m)$$

and using (37) we may write relations (40) in the form

$$s_k = \int_{-\infty}^{\infty} x^k f_L^*(x)\,dx \qquad (k = 0, 1, 2, \cdots, 2m - 1), \tag{41}$$

where

$$f_L^*(x) = \frac{L}{2}\left\{1 - \operatorname{sign}\frac{\psi(x)}{\phi(x)}\right\}.$$

Formulas (41) show that the sequence

$$s_0, s_1, \cdots, s_{2m-2} \tag{42}$$

is positive definite (or negative definite in the interval $(-\infty, \infty)$ if $L > 0$ (or $L < 0$).

In addition on the basis of Theorem 8 we conclude that $|L| \geq \lambda_{m-1}$.

We turn now to the proof of sufficiency.

We shall assume that the sequence (42) is positive definite (otherwise substitute $q(x)$ for $-q(x)$).

Having selected an arbitrary number $L > \lambda_{m-1}$ we may on the basis of Theorem 7 find a function

$$f_L^*(x) = \frac{L}{2}\left\{1 - \operatorname{sign}\frac{\psi_m(x)}{\phi_m(x)}\right\},$$

for which

$$s_k = \int_{-\infty}^{\infty} x^k f_L^*(x)\,dx \qquad (k = 0, 1, 2, \cdots, 2m - 1),$$

where $\phi_m(x)$ and $\psi_m(x)$ are polynomials of degree m with alternating roots.

By reversing the reasoning used in the proof of necessity we may easily verify that the polynomials $\phi_m(x)$ and $\psi_m(x)$ generate a quadrature formula of order $2m$ of A. A. Markov.

Remark. It follows from the proof that a function $q(x)$ admitting one quadrature formula of A. A. Markov admits an infinite number of them: one for each L satisfying the inequality

$$L > \lambda_{m-1} \quad (\text{or } L < -\lambda_{m-1}),$$

if $\int_{-\infty}^{\infty} xq(x)\,dx > 0$ (or $\int_{-\infty}^{\infty} xq(x)\,dx < 0$); we see also that the methods of construc-

tion of A. A. Markov's formulas which we have indicated give all formulas of order $2m$, except perhaps those for which $|L| = \lambda_{m-1}$.

Theorem 13. *Let the sequence*

$$s_0, s_1, \cdots, s_{2m-2} \tag{42}$$

be positive definite in the interval $(-\infty, \infty)$.

a) *If the sequence* (42) *is complete, then the function* $q(x)$ *admits a quadrature formula of P. Čebyšev of order* $2m - 1$.

b) *If the sequence* (42) *is incomplete, then the function* $q(x)$ *admits a quadrature formula of P. Čebyšev with the number of nodes* $\leq 2m - 4$ *valid for polynomials of degree* $\leq 2m - 2$.

The proof will be omitted, as it follows in the same manner as the proof of sufficiency in Theorem 12, if we use the results of §3 of this chapter. We shall only remark that the formulas of P. Čebyšev whose existence is given by Theorem 13 are at the same time formulas of A. A. Markov, i.e., in these formulas the nodes α_i and the nodes β_i do alternate.

We also note that in case b) we may say that $q(x)$ admits the above Čebyšev formula of order $2m - 2$.

3. We turn now to the case of a finite interval. Assuming that the function $q(x)$ satisfies relation (31) we shall consider quadrature formulas of the type of (32).

As before, we shall call the number n the order of the quadrature formula (32) if formula (32) is true for all polynomials of degree $\leq n$ but is not true for all polynomials of degree $n + 1$.

In addition, we shall agree hereafter to denote by r the number of nodes α_i and β_i which lie within the interval $-1 < x < 1$, and shall therefore, in calculating r, omit those nodes which coincide with the endpoints of the interval $< -1, 1>$.

Modifying slightly our earlier definitions we shall call formula (32) a Čebyšev formula if $n \geq r + 1$ and a Markov formula if $n = r$ and if, moreover, the nodes α_i alternate with the nodes β_i.

Retaining the earlier notation

$$(k+1)s_k = \int_{-1}^{1} x^{k+1} q(x)\,dx \qquad (k = 0, 1, 2, \cdots),$$

we shall prove the following proposition.

Theorem 14. *In order that the function $q(x)$ admit the quadrature formula of A. A. Markov*

$$\int_{-1}^{1} F(x)q(x)dx = 2L \sum_{i=1}^{m} \{F(\xi_i) - F(\eta_i)\} \tag{43}$$

of order n, it is necessary and sufficient that the sequence

$$s_0, s_1, \cdots, s_{n-1} \tag{44}$$

be positive definite or negative definite in the interval $<-1, 1>$.

If this condition is fulfilled, then there exist exactly two formulas of type (43). *These formulas are characterized by the following arrangements of nodes:*

$$-1 < \eta_1 < \xi_1 < \cdots < \eta_m < \xi_m < 1 \qquad (m = \tfrac{n}{2}),$$

$$-1 = \eta_1 < \xi_1 < \cdots < \eta_m < \xi_m = 1 \qquad (m = \tfrac{n}{2} + 1),$$

if n is an even number; and

$$\left.\begin{aligned} &-1 < \eta_1 < \xi_1 < \cdots < \eta_m < \xi_m = 1 \\ &-1 = \eta_1 < \xi_1 < \cdots < \eta_m < \xi_m < 1 \end{aligned}\right\} \quad (m = \tfrac{n+1}{2}),$$

if n is an odd number.

Proof. The validity of the Markov formulas (43) for any polynomial of degree n is equivalent to the correctness of the equalities

$$s_k = \frac{2L}{k+1} \sum_{i=1}^{m} (\xi_i^{k+1} - \eta_i^{k+1}) = 2L \sum_{i=1}^{m} \int_{\eta_i}^{\xi_i} x^k dx =$$

$$= L \int_{-1}^{1} x^k \left\{1 - \operatorname{sign} \frac{\psi(x)}{\phi(x)}\right\} dx = \int_{-1}^{1} x^k f_L^*(x)dx \qquad (k = 0, 1, \cdots, n-1),$$

where

$$\phi(x) = (x - \xi_1) \cdots (x - \xi_m),$$

$$\psi(x) = (x - \eta_1) \cdots (x - \eta_m).$$

Therefore Theorem 14 is a direct result of assertion d) of Theorem 11.

We further note that it follows from assertion c) of Theorem 11 that

Theorem 15. *If a sequence*

$$s_0, s_1, \cdots, s_{n-1} \tag{44}$$

is positive [1] *definite in the interval* $<-1, 1>$ *then the function* $q(x)$ *admits a P. Čebyšev formula of order* n.

Remark 1. Introducing a number θ which satisfies the inequality $-1 < \theta < 1$, we may construct quadrature formulas of the type

$$\int_{-1}^{1} F(x)q(x)dx = M\left\{-\frac{1+\theta}{2}F(-1) + \sum_{i=1}^{m+1} F(\alpha_i) - \sum_{i=1}^{m} F(\beta_i) - \frac{1-\theta}{2}F(1)\right\}.$$

For these the theory which has been developed remains valid and Theorems 14 and 15 are even simplified because the condition of positive or negative definiteness in the interval $<-1, 1>$ of sequence (44) drops out.

Remark 2. P. Čebyšev illustrated his rule for the functions

$$q(x) = x, \quad \frac{x}{\sqrt{1-x^2}} \qquad (-1 \leq x \leq 1).$$

In the first case $(q(x) = x)$ we find that

$$(k+1)s_k = \int_{-1}^{1} x^{k+1}xdx = (k+1)\int_{-1}^{1} x^k \frac{1-x^2}{2}dx,$$

and the sequence

$$s_k = \frac{1}{2}\int_{-1}^{1} x^k(1-x^2)dx \qquad (k = 0, 1, 2, \cdots)$$

is positive definite in the interval $<-1, 1>$.

In the second case $\left(q(x) = \frac{x}{\sqrt{1-x^2}}\right)$ we also find that the sequence

$$s_0, s_1, s_2, \cdots$$

is positive definite in the interval $<-1, 1>$, because

$$(k+1)s_k = \int_{-1}^{1} x^{k+1}\frac{x}{\sqrt{1-x^2}}dx = (k+1)\int_{-1}^{1} x^k\sqrt{1-x^2}\,dx \qquad (k = 0, 1, 2, \cdots).$$

Therefore Theorems 14 and 15 are applicable to the functions x and $\frac{x}{\sqrt{1-x^2}}$.

If we set

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1) The theorem obviously remains valid if sequence (44) is negative definite in the interval $<-1, 1>$.

$$q_1(x) = \begin{cases} 0 & -\infty < x < 1 \\ x & -1 \le x \le 1 \\ 0 & 1 < x < \infty \end{cases} \qquad q_2(x) = \begin{cases} 0 & -\infty < x \le 1 \\ \dfrac{x}{\sqrt{1-x^2}} & -1 < x < 1 \\ 0 & 1 \le x < \infty, \end{cases}$$

we find that Theorems 12 and 13 are also applicable to $q_1(x)$ and $q_2(x)$.

It is easy to see that in general if $q(x) \cdot x^{2m}$ is absolutely integrable in the interval $(-\infty, \infty)$ and if the function

$$p(x) = \int_{-\infty}^{x} q(x)\,dx$$

has a constant sign in the interval $(-\infty, \infty)$, then Theorems 12 and 13 will be applicable to $q(x)$.

In fact in this case

$$s_j = \int_{-\infty}^{\infty} \frac{1}{j+1} x^{j+1} q(x)\,dx = \frac{1}{j+1} \int_{-\infty}^{\infty} x^{j+1}\,dp(x) =$$

$$= \lim_{N\to\infty} \left\{ -\int_{-N}^{N} x^j p(x)\,dx + \frac{x^{j+1}}{j+1} p(x) \Big|_{-N}^{N} \right\}.$$

Because

$$N^{j+1} p(N) = N^{j+1} \int_{-\infty}^{N} q(x)\,dx = -N^{j+1} \int_{N}^{\infty} q(x)\,dx,$$

therefore

$$|N^{j+1} p(N)| \le \int_{N}^{\infty} x^{j+1} |q(x)|\,dx \le \int_{N}^{\infty} x^{2m} |q(x)|\,dx \to 0 \qquad (N \to \infty)$$

and in a similar way

$$N^{j+1} p(-N) \to 0.$$

Thus,

$$s_j = -\int_{-\infty}^{\infty} x^j p(x)\,dx \qquad (j = 0, 1, \cdots, 2m-2),$$

from which it follows that the sequence

$$s_0, s_1, \cdots, s_{2m-2}$$

is positive definite in the interval $(-\infty, \infty)$ for $p(x) \le 0$ and negative definite for $p(x) \ge 0$.

We shall give several examples.

1. Formulas of A. A. Markov

$$\int_{-1}^{1} x F(x)\,dx = \frac{1}{2}\left\{F\left(\frac{2}{3}\right) - F\left(-\frac{2}{3}\right)\right\} + \frac{7}{405} F'''(\xi),$$

$$\int_{-1}^{1} x F(x)\,dx = \frac{1}{2}\left\{F\left(\frac{1}{3} + \sqrt{\frac{22}{135}}\right) - F\left(-\frac{1}{3} + \sqrt{\frac{22}{135}}\right) + \right.$$

$$\left. + F\left(\frac{1}{3} - \sqrt{\frac{22}{135}}\right) - F\left(-\frac{1}{3} - \sqrt{\frac{22}{135}}\right)\right\} + \frac{1306}{2296350} F^{(V)}(\eta).$$

2. Formulas of P. L. Čebyšev

$$\int_{-1}^{1} xF(x)\,dx = \sqrt{\frac{5}{27}}\left\{F\left(\sqrt{\frac{3}{5}}\right) - F\left(-\sqrt{\frac{3}{5}}\right)\right\},$$

$$\int_{-1}^{1} x F(x)\,dx = 0{,}23937\,\{F(0{,}89224) + F(0{,}50030) - F(-0{,}50030) -$$

$$- F(-0{,}89224)\}.$$

4. The foregoing considerations may also be applied to the construction of quadrature formulas of the type [1]

$$\int_{-1}^{1} F(x) g(x)\,dx = G\left\{\sum_{i=1}^{m+1} F(\xi_1) - \sum_{i=1}^{m} F(\eta_i)\right\}, \tag{45}$$

valid for any polynomial $F(x)$ of degree $\leq 2m + 1$, where

$$G = \int_{-1}^{1} g(x)\,dx > 0. \tag{46}$$

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1) A. A. Markov also first concerned himself with similar formulas. They represent particular cases of formulas of order n of the form

$$\int_{-1}^{1} F(x) g(x)\,dx = C \sum_{i=1}^{n} \epsilon_i F(x_i) \quad (\epsilon_i = \pm 1,\ -1 \leq x_i \leq 1). \tag{α}$$

The last formula contains as a particular case ($\epsilon_i = 1$) the quadrature formula of P. L. Čebyšev [18b]

$$\int_{-1}^{1} F(x) g(x)\,dx = \frac{G}{n} \sum_{i=1}^{n} F(x_i), \tag{β}$$

where

$$-1 \leq x_i \leq 1, \quad G = \int_{-1}^{1} g(x)\,dx > 0.$$

Theorem 16. *For the function $g(x)$ to satisfy quadrature formula (45) of order $2m+1$ with nodes ξ_i, η_i, which satisfy the inequality*

$$-1 \le \xi_1 \le \eta_1 \le \xi_2 \le \cdots \le \xi_{m+1} \le 1,$$

it is necessary and sufficient that the sequence $\sigma_0 = 1, \sigma_1, \cdots, \sigma_{2m+1}$ be non-negative definite in the interval $<-1, 1>$ having been defined by the expansion

$$e^{\frac{1}{G}\left\{\frac{s_0'}{z}+\frac{s_1'}{z^2}+\cdots+\frac{s_{2m}'}{z^{2m+1}}\right\}} = \sigma_0 + \frac{\sigma_1}{z} + \cdots + \frac{\sigma_{2m+1}}{z^{2m+1}} + \cdots, \tag{47}$$

where

$$s_k' = \frac{1}{k+1}\int_{-1}^{1} x^{k+1} g(x)\,dx \qquad (k = 0, 1, 2, \cdots). \tag{47$_1$}$$

Proof. At the start we remark that relation (47) may be written in the form

$$\frac{1}{z-1}\, e^{\frac{1}{G}\left\{\frac{s_0}{z}+\frac{s_1}{z^2}+\cdots+\frac{s_{2m}}{z^{2m+1}}\right\}} = \frac{\sigma_0}{z} + \frac{\sigma_1}{z^2} + \cdots + \frac{\sigma_{2m+1}}{z^{2m+2}} + \cdots, \tag{48}$$

where

$$s_k = \frac{1}{k+1}\left\{-G + \int_{-1}^{1} x^{k+1} g(x)\,dx\right\}. \tag{48$_1$}$$

We shall prove the necessity first.

Suppose the function $g(x)$ admits quadrature formula (45), where

$$-1 \le \xi_1 \le \eta_1 \le \xi_2 \le \cdots \le \xi_{m+1} \le 1.$$

It follows from (45) and (48$_1$) that

$$s_k = \frac{1}{k+1} G\{\xi_1^{k+1} - \eta_1^{k+1} + \ldots + \xi_{m+1}^{k+1} - 1\} =$$

$$= -\frac{1}{2} G \int_{-1}^{1} x^k \left\{1 - \operatorname{sign} \frac{(x-\eta_1)(x-\eta_2)\ldots(x-\eta_m)(x-1)}{(x-\xi_1)(x-\xi_2)\ldots(x-\xi_m)(x-\xi_{m+1})}\right\} dx =$$

$$= \int_{-1}^{1} x^k f_G(x)\,dx \qquad (k = 0, 1, \ldots, 2m),$$

where

$$-G \le f_G(x) \le 0.$$

From this on the basis of the note to Theorem 5 (if we set $\theta = -1$, $2L = G$) we find that the sequence

$$\sigma_0, \sigma_1, \cdots, \sigma_{2m+1} \tag{49}$$

is non-negative definite in the interval $<-1, 1>$.

We turn to the proof of the sufficiency. Let sequence (49) be non-negative definite in the interval $<-1, 1>$. From Theorem 16a of Chapter I there is a uniquely determined representation

$$\sigma_k = \sum_{i=1}^{p} \rho_i \xi_i^k \qquad (k = 0, 1, \cdots, 2m+1),$$

where $p \leq m+1$, $\rho_i > 0$ and

$$-1 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq 1.$$

Thus, there exist numbers η_i such that

$$\xi_1 < \eta_1 < \xi_2 < \cdots < \xi_{p-1} < \eta_{p-1} < \xi_p$$

and

$$\frac{(z-\eta_1)\cdots(z-\eta_{p-1})}{(z-\xi_1)(z-\xi_2)\cdots(z-\xi_p)} = \frac{\sigma_0}{z} + \frac{\sigma_1}{z^2} + \cdots + \frac{\sigma_{2m+1}}{z^{2m+2}} + \cdots.$$

Therefore by formula (48)

$$e^{\frac{1}{G}\left\{\frac{s_0}{z} + \cdots \frac{s_{2m}}{z^{2m+1}}\right\}} = \frac{(z-\eta_1)\cdots(z-\eta_{p-1})(z-1)}{(z-\xi_1)\cdots(z-\xi_{p-1})(z-\xi_p)} + \frac{1}{z^{2m+2}}\,\mathfrak{P}\left(\frac{1}{z}\right).$$

From this, by repeating the argument which we have already used many times, we obtain

$$\frac{s_0}{z} + \cdots + \frac{s_{2m}}{z^{2m+1}} + \cdots = -\frac{1}{2}\,G \int_{-1}^{1} \frac{1}{z-x}\left\{1 - \operatorname{sign}\frac{(x-\eta_1)\cdots(x-1)}{(x-\xi_1)\cdots(x-\xi_p)}\right\}dx,$$

so that

$$s_k = -\frac{1}{2}\,G \int_{-1}^{1} x^k \left\{1 - \operatorname{sign}\frac{(x-\eta_1)\cdots(x-1)}{(x-\xi_1)\cdots(x-\xi_p)}\right\}dx = \frac{G}{k+1}\left\{\xi_1^{k+1} - \eta_1^{k+1} + \cdots + \xi_p^{k+1} - 1\right\}$$

and thus by formula (48_1)

$$\int_{-1}^{1} x^{k+1} g(x)\,dx = G\left\{\sum_{i=0}^{p} \xi_i^{k+1} - \sum_{i=1}^{p-1} \eta_i^{k+1}\right\} \qquad (k = 0, 1, \cdots, 2m).$$

Thus, quadrature formula (45) is constructed and the theorem is proved in its entirety.

Example. Let $g(x) = 1$ $(-1 \leq x \leq 1)$.

In accordance with formula (48_1)

$$s_k = \frac{1}{k+1}\left\{-2 + \int_{-1}^{1} x^{k+1}\,dx\right\} = -\int_{-1}^{1} x^k(1+x)\,dx \qquad (k = 0, 1, \cdots),$$

and since

$$-2 \leq -(1+x) \leq 0 \qquad (-1 \leq x \leq 1),$$

by the remark to Theorem 5 (if we set $\theta = -1$ and $L = 1$) the sequence determined with the aid of expansion (48),

$$\sigma_0, \sigma_1, \cdots, \sigma_n$$

is positive definite in the interval $<-1, 1>$ for any n.

It then follows that for any m there exists a quadrature formula of order $2m+1$

$$\int_{-1}^{1} F(x)\,dx = 2\left\{\sum_{i=1}^{m+1} F(\xi_i) - \sum_{i=1}^{m} F(\eta_i)\right\},$$

where

$$-1 < \xi_1 < \eta_1 < \cdots < \xi_{m+1} < 1.$$

In particular, the following formulas found by A. A. Markov [13b] are valid:

$$\int_{-1}^{1} F(x)\,dx = 2F(0) + \frac{1}{3}F''(\zeta_0),$$

$$\int_{-1}^{1} F(x)\,dx = 2\left\{F\left(\sqrt{\frac{1}{6}}\right) - F(0) + F\left(-\sqrt{\frac{1}{6}}\right)\right\} + \frac{13}{1080}F^{(IV)}(\zeta_1),$$

$$\int_{-1}^{1} F(x)\,dx = 2\left\{F\left(\sqrt{\frac{23}{60}}\right) - F\left(\sqrt{\frac{13}{60}}\right) + F(0) - F\left(-\sqrt{\frac{13}{60}}\right) + F\left(-\sqrt{\frac{23}{60}}\right)\right\} + \frac{3821}{360 \cdot 75600}F^{(VI)}(\zeta_2).$$

The following theorem in a certain sense complements Theorem 16. [1)]

Theorem 17. *If the quadrature formula* (45), (46) *is valid for any polynomial of degree* $\leq 2m+1$ *and if the nodes* ξ_i *and* η_i *do not satisfy the inequality*

$$-1 < \xi_1 \leq \eta_1 \leq \cdots \leq \xi_{m+1} < 1,$$

1) This proposition was kindly pointed out to the authors by Academician S. N. Bernštein.

then it is impossible for the sequence

$$s_k'' = \int_{-1}^{1} x^k g(x)\,dx \qquad (k = 0, 1, \cdots, 2m+1)$$

to be non-negative definite in the interval $<-1, 1>$.

Proof. Let the quadrature formula (45), (46) be valid for some polynomial of degree $\leq 2m+1$ and let the sequence s_k'' $(k = 0, 1, \cdots, 2m+1)$ be non-negative definite in the interval $<-1, 1>$.

Without loss of generality we may agree that none of the nodes η_i shall coincide with any one of the nodes ξ_j, and we must therefore prove that first the nodes η_i alternate with the nodes ξ_j and secondly $\xi_1 > -1$ and $\xi_{m+1} < 1$, the nodes ξ_j and η_i being numbered in order of increasing value.

We shall begin with the proof that they alternate. Assume that between the nodes ξ_k and ξ_{k+1} there does not lie one of the nodes η_j.

Let

$$\eta_1 < \eta_2 < \cdots < \eta_i < \xi_k; \quad \xi_{k+1} < \eta_{i+1} < \cdots < \eta_m.$$

Clearly either $i \geq k$ or $m - i \geq m + 1 - k$. Let us assume for definiteness that $i \geq k$.

We construct a polynomial $F(x)$ of lowest possible degree satisfying the conditions

$$\begin{aligned}
&F(\xi_1) = 1, \quad F'(\xi_1) = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&F(\xi_{k-1}) = 1, \quad F'(\xi_{k-1}) = 0\\
&F(\xi_k) = 1\\
&F(\xi_{k+1}) = F'(\xi_{k+1}) = 0.\\
&\cdots\cdots\cdots\cdots\cdots\\
&F(\xi_{m+1}) = F'(\xi_{m+1}) = 0.
\end{aligned}$$

Because we are given $2m+1$ conditions for determining the polynomial $F(x)$, its degree does not exceed $2m$.

The derivative of $F(x)$ has the zeros $\xi_1, \xi_2, \cdots, \xi_{k-1}, \xi_{k+1}, \cdots, \xi_{m+1}$ by construction, and moreover it must have at least one zero in each of the open intervals

$$(\xi_1, \xi_2),\ (\xi_2, \xi_3), \cdots, (\xi_{k-1}, \xi_k),\ (\xi_{k+1}, \xi_{k+2}), \cdots, (\xi_m, \xi_{m+1})$$

by Rolle's theorem, i.e., at least $m-1$ more zeros.

It is therefore apparent that $F'(x)$ has exactly $2m-1$ zeros (which means $F(x)$ is a polynomial of exactly $2m$th degree) so that in the interval (ξ_k, ξ_{k+1}) the derivative $F'(x)$ does not vanish and in each of the remaining intervals (ξ_i, ξ_{i+1}) there is exactly one root ζ_i.

Since $F(\xi_k) = 1$, $F(\xi_{k+1}) = 0$, in the interval (ξ_k, ξ_{k+1}) the polynomial $F(x)$ decreases, remaining non-negative. Therefore it follows that at the point ξ_{k+1} we have a minimum, at the point ζ_{k+1} a maximum, etc. In a similar way we find that at the point ζ_{k-1} there is a maximum and at the point ξ_{k-1} a minimum, etc. These considerations show that the polynomial $F(x)$ is non-negative on the entire axis and, moreover, for $x \leq \xi_k$ it satisfies the inequality $F(x) \geq 1$. We suggest that the reader construct a graph of the polynomial $F(x)$ in order better to follow the properties we have asserted for it.

We now apply formula (45) to $F(x)$.

Since the polynomial $F(x)$ is everywhere non-negative because of the non-negativeness of the sequence $s_0'', s_1'', \cdots, s_{2m+1}''$ we have the inequality

$$\int_{-1}^{1} F(x)\, g(x)\, dx \geq 0.$$

Now consider the right member of the integrand. By choice of $F(x)$ it is equal to

$$G\left\{k - \sum_{j=1}^{m} F(\eta_j)\right\} < G\left\{k - \sum_{j=1}^{i} F(\eta_j)\right\} < 0,$$

because $i \geq k$ and $F(\eta_j) > 1$ for $j = 1, 2, \cdots, i$.

The result we have obtained in this fashion is absurd, which proves the nodes alternate.

We now turn to the second part of the theorem.

Assume that $\xi_{m+1} \geq 1$.

We take the polynomial

$$F(x) = (x - \xi_1)^2 \cdots (x - \xi_m)^2 (1 - x).$$

Quadrature formula (45) is valid for it. But since $F(x) \geq 0$ for $x \leq 1$, by the non-negativeness of the sequence $s_0'', s_1'', \cdots, s_{2m+1}''$ we have the inequality

$$\int_{-1}^{1} F(x)\, g(x)\, dx \geq 0.$$

On the other hand, the right side of formula (45) equals

$$G\left\{F(\xi_{m+1}) - \sum_{i=1}^{m} F(\eta_i)\right\} < 0$$

in as much as $F(\xi_{m+1}) \leq 0$ and $F(\eta_i) > 0$ $(i = 1, 2, \cdots, m)$. It is analogously proved that $\xi_1 > -1$.

GENERAL THEOREMS ABOUT POSITIVE DEFINITE FUNCTIONALS

M. KREĬN

In the first article of this book we gave the solution to a set of transcendental problems (exponential and trigonometric moment problems) with the help of a transition, based on the theorems of Helly, from the corresponding algebraic problems. However, in 1921 Marcel Riesz [37] proposed another method which allows us to solve the indicated transcendental problems quickly and directly (without limit processes). This method of M. Riesz is noteworthy, moreover, in that it allows far-reaching generalizations, as Riesz himself noted.

The present article is devoted to an exposition and development of Riesz's clever idea.

§ 1

1. Let E be a certain collection of complex functions which are defined for all values t of some set I. A functional $\mathfrak{P}(\phi)$, which relates a certain complex number to every function $\phi \subset E$ will be called *non-negative definite* if to any non-negative combination

$$\sum_1^n \alpha_k \phi_k(t) + \sum_1^n \bar{\alpha}_k \bar{\phi}_k(t) \geq 0 \qquad (t \subset I), \tag{1}$$

where $\phi_k \subset E$ $(k = 1, 2, \cdots, n)$ and α_k $(k = 1, 2, \cdots, n)$ are arbitrary complex numbers, there corresponds the non-negative number

$$\sum_1^n \alpha_k \mathfrak{P}(\phi_k) + \sum_1^n \bar{\alpha}_k \overline{\mathfrak{P}(\phi_k)} \geq 0. \tag{2}$$

In particular, if in (1) equality holds, then it will in (2) also. Therefore if $\mathfrak{P}$ is a non-negative definite functional, and

$$\phi = \sum_1^n \alpha_k \phi_k + \sum_1^n \bar{\alpha}_k \bar{\phi}_k, \tag{3}$$

where $\phi_k \subset E$ $(k = 1, 2, \cdots, n)$, and α_k $(k = 1, 2, \cdots, n)$ are arbitrary complex numbers, we set

$$\mathfrak{P}(\phi) = \sum_1^n \alpha_k \mathfrak{P}(\phi_k) + \sum_1^n \bar{\alpha}_k \overline{\mathfrak{P}(\phi_k)},$$

we thereby uniquely determine a real functional $\mathfrak{P}$ on the collection $\breve{E}$ of all real functions $\phi(t)$ of the form of (3). If the initial collection E consists only of real functions, then $\breve{E}$ is the real linear envelope of the collection E; in the general case $\breve{E}$ consists of a linear envelope of the set of real and imaginary parts of the functions of the set E. The functional $\mathfrak{P}(\phi)$ in $\breve{E}$ is obviously distributive, i.e.,

$$\mathfrak{P}(\phi' + \phi'') = \mathfrak{P}(\phi') + \mathfrak{P}(\phi'') \quad \text{for} \quad \phi', \phi'' \subset \breve{E},$$

$$\mathfrak{P}(\lambda\phi) = \lambda\mathfrak{P}(\phi) \text{ if } \phi \subset \breve{E} \text{ and } \lambda \text{ is a real number.}$$

The functional $\mathfrak{P}(\phi)$, being non-negative definite in E, is also non-negative definite in $\breve{E}$, which last simply means that

$$\mathfrak{P}(\phi) \geq 0 \quad \text{if} \quad \phi(t) \geq 0 \qquad (\phi \subset \breve{E}), \tag{4}$$

or equivalently that

$$\mathfrak{P}(\phi') \leq \mathfrak{P}(\phi'') \quad \text{if} \quad \phi'(t) \leq \phi''(t) \qquad (\phi', \phi'' \subset E).$$

A functional $\mathfrak{P}(\phi)$ which is non-negative definite in E will be called *simply positive definite* if in (4) $\mathfrak{P}(\phi) > 0$ in every case where $\phi(t)$ is not identically zero.

Let $\psi(t)$ be a real function defined in I. We shall say that it is *dominated* by the collection E if in $\breve{E}$ we can find two functions $\phi'(t)$ and $\phi''(t)$ such that

$$\phi'(t) \leq \psi(t) \leq \phi''(t) \qquad (t \subset I).$$

Lemma. *If F is a denumerable* [1] *set of functions which are dominated by E, then every non-negative definite functional $\mathfrak{P}$ in E may be extended to the elements in F in such a manner as to be non-negative definite on the set $E + F$.*

Proof. Let $F = \{\psi_i\}_1^\infty$. Consider all possible functions ϕ' and $\phi'' \subset \breve{E}$ such that the inequality

$$\phi' \leq \psi_1 \leq \phi''$$

is satisfied. Since the functional $\mathfrak{P}(\phi)$ is non-negative definite $\mathfrak{P}(\phi') \leq \mathfrak{P}(\phi'')$, and hence there exists a number ξ such that

$$\sup \mathfrak{P}(\phi') \leq \xi \leq \inf \mathfrak{P}(\phi''). \tag{5}$$

We set

$$\mathfrak{P}(\psi_1) = \xi, \quad E_1 = E + \{\psi_1\}.$$

---

1) Use of Zermelo's well-ordering theorem would enable us to dispense with this condition of denumerability.

The functional $\mathfrak{P}$ is now defined in E_1; we shall show that it is non-negative definite in E_1. Indeed, if

$$\chi = \phi + c\,\psi_1 \geq 0 \qquad (\phi \subset \breve{E}),$$

then, assuming $c > 0$, we have

$$-\frac{1}{c}\,\phi \leq \psi_1$$

and so by the first of the inequalities of (5)

$$-\frac{1}{c}\,\mathfrak{P}(\phi) \leq \mathfrak{P}(\psi_1)$$

or

$$\mathfrak{P}(\chi) = \mathfrak{P}(\phi) + c\,\mathfrak{P}(\psi_1) \geq 0. \tag{6}$$

This inequality is similarly demonstrated for $c < 0$. In the case $c = 0$ inequality (6) is valid because of the non-negativeness of $\mathfrak{P}$ in E.

Thus the functional $\mathfrak{P}$ is defined in E_1 and is non-negative definite there.

After this we may define $\mathfrak{P}(\psi_2)$ so that the functional $\mathfrak{P}$ is non-negative definite in

$$E_2 = E_1 + \{\psi_2\} = E + \{\psi_1, \psi_2\}.$$

Continuing this process to infinity we may define the functional $\mathfrak{P}$ on the whole set

$$E + F = E + \{\psi_1, \psi_2, \cdots\}$$

so that it is non-negative definite on this set.

2. We now consider the case where I is the closed finite interval $<a, b>$ and every function $\phi \subset E$ is continuous in I. For this case it is true that

Theorem 1. *Let the set $\breve{E}$ contain at least one positive function $\omega_0(t)$ in $I = <a, b>$. It is then necessary and sufficient for a given functional $\mathfrak{P}$ defined in E to admit a representation*

$$\mathfrak{P}(\phi) = \int_a^b \phi(t)\,d\sigma(t) \qquad (\phi \subset E), \tag{7}$$

where $\sigma(t)$ is some function which is non-decreasing in $<a, b>$, that the functional be non-negative definite.

Proof. The necessity of the condition is trivial, since if $\mathfrak{P}(\phi)$ admits a representation (7) in E, then it also admits such a representation in $\breve{E}$; on the other hand,

$$\int_a^b \phi(t)\,d\sigma(t) \geq 0 \quad \text{for} \quad \phi(t) \geq 0 \qquad (a \leq t \leq b).$$

We now prove sufficiency of the condition.

Let α be some arbitrary point selected from the interior of the interval I. [1] For $\tau > \alpha$ we set

$$\zeta_\tau(t) = \begin{cases} 1 & \text{for} \quad \alpha \leq t \leq \tau, \\ 0 & \text{elsewhere}, \end{cases} \tag{8}$$

and for $\tau < \alpha$

$$\zeta_\tau(t) = \begin{cases} -1 & \text{for} \quad \tau \leq t < \alpha, \\ 0 & \text{otherwise}. \end{cases} \tag{9}$$

Let $\{t_k\}_1^\infty$ be some sequence of points dense in I containing the endpoints a and b. Since for sufficiently large C_k $(k = 1, 2, \cdots)$

$$-C_k \omega_0(t) < \xi_{t_k}(t) < C_k \omega_0(t) \text{ and } \pm C_k \omega_0(t) \subset \breve{E} \qquad (k = 1, 2, \cdots),$$

by the lemma § 1 we may extend the functional $\mathfrak{P}$ on all of the functions $\xi_{t_k}(t)$ $(k = 1, 2, \cdots)$ so that it is positive definite on the collection $E + \{\xi_{t_k}(t)\}_1^\infty$. From the inequality

$$\phi + \sum_1^n c_k \xi_{t_k}(t) \geq 0 \qquad (\phi \subset \breve{E}),$$

(where n is any integer) there follows the inequality

$$\mathfrak{P}(\phi) + \sum_1^n c_k \mathfrak{P}(\xi_{t_k}) \geq 0.$$

Hereafter we set

$$\sigma(t_k) = \mathfrak{P}(\xi_{t_k}) \qquad (k = 1, 2, \cdots). \tag{10}$$

Since

$$\xi_{t_j}(t) < \xi_{t_k}(t) \quad (t \subset I) \quad \text{for} \quad t_j < t_k,$$

then

$$\sigma(t_j) \leq \sigma(t_k) \quad \text{for} \quad t_j < t_k.$$

The function $\sigma(t)$ will be extended to all other points $t \neq t_k$ $(k = 1, 2, \cdots)$ on the interval I so that it remains monotonic, by setting for example

$$\sigma(t) = \sup_{t_k < t} \sigma(t_k) \qquad (t \subset I), \tag{10'}$$

---

1) The requirement that the point α lies within I is not really essential; we make it only so that we may transfer most of our reasoning in Theorem 1 to the proof of Theorem 4 (cf. p. 137).

and we shall show that the construction of the function in such a manner gives the desired representation of the functional $\mathfrak{P}(\phi)$. Because the real and imaginary parts of a function $\phi \subset E$ always belong to $\breve{E}$, equality (7) will be proved if we prove that

$$\mathfrak{P}(\phi) = \int_a^b \phi(t)\, d\sigma(t) \quad \text{for} \quad \phi \subset \breve{E}.$$

Let $\phi \subset \breve{E}$. Then for any $\epsilon > 0$ we may always determine the points $a = s_1 < s_2 < \cdots < s_n = b$ belonging to the sequence $\{t_k\}_1^\infty$ so that

$$-\epsilon\omega(t) < \phi(t) - \sum_1^{n-1} \phi(s_k)[\xi_{s_{k+1}}(t) - \xi_{s_k}(t)] < \epsilon\omega(t).$$

Applying the functional $\mathfrak{P}$ to both sides, we have

$$-\epsilon\,\mathfrak{P}(\omega) < \mathfrak{P}(\phi) - \sum_{k=1}^{n-1} \phi(s_k)[\sigma(s_{k+1}) - \sigma(s_k)] < \epsilon\,\mathfrak{P}(\omega).$$

Taking the limit when $\max(s_{k+1} - s_k) \to 0$ we have, moreover,

$$\left| \mathfrak{P}(\phi) - \int_a^b \phi(t)\, d\sigma(t) \right| < \epsilon\,\mathfrak{P}(\omega).$$

From the last inequality (7) follows because of the arbitrariness of $\epsilon > 0$. The theorem is proved.

3. Let

$$w_0(t),\ w_1(t),\ w_2(t), \cdots \qquad (t \subset I)$$

be some sequence of complex functions defined in I.

We shall call a sequence of complex numbers $c_0, c_1, c_2, \cdots$ *non-negative definite (positive definite) relative to the sequence of functions* $\{w_k(t)\}$ if the functional $\mathfrak{P}(\phi)$ defined on the sequence $\{w_k\}$ by the equalities

$$\mathfrak{P}(w_k) = c_k \qquad (k = 0, 1, \cdots)$$

is non-negative definite (or positive definite).

It is obvious that the definition of definite non-negativeness (or definite positiveness) of the sequence $\{c_k\}$ is an immediate generalization of that cited at the beginning of Chapter I of Article I.

As an immediate consequence of Theorem 1 we have

Theorem 2. [1] *Let the sequence* $\{w_k(t)\}$ *consist of continuous functions in the finite interval* $<a, b>$ *with the property that we may form at least one positive polynomial:*

$$\omega(t) = \sum_1^m \alpha_k w_k(t) + \sum_1^m \bar{\alpha}_k \bar{w}_k(t) > 0 \qquad (a \leq t \leq b).$$

Then in order that to the sequence of numbers $\{c_k\}$ *there correspond at least one non-decreasing function* $\sigma(t)$ $(a \leq t \leq b)$ *such that*

$$c_k = \int_a^b w_k(t)\, d\sigma(t) \qquad (k = 0, 1, \cdots), \tag{11}$$

it is necessary and sufficient that the sequence $\{c_k\}$ *be non-negative definite relative to the sequence* $\{w_k(t)\}$.

Proof. The set $E = \{w_k(t)\}$ does in fact satisfy the conditions of the previous theorem ($\breve{E}$ contains the positive function $\omega(t)$). On the other hand, the non-negativeness of the sequence $\{c_k\}$ relative to the sequence $\{w_k(t)\}$ is equivalent, by definition, to the positiveness of the functional $\mathfrak{P}$ defined on E by the equalities

$$\mathfrak{P}(w_k) = c_k \qquad (k = 0, 1, \cdots).$$

This proves the theorem.

It will later be our goal to show that it is possible to prove the theorem by certain geometric ideas. Without loss of generality we may consider all the functions $w_k(t)$ to be real, since otherwise we could turn from the sequence $\{w_k(t)\}$ to the sequences composed of the real and imaginary parts of the functions $w_k(t)$ and then similarly transform the sequence of numbers $\{c_k\}$.

Moreover, we shall initially suppose that the sequence $\{w_k\}$, and thus the sequence $\{c_k\}$, has a finite number of terms, say n.

We shall denote by $\mathfrak{R}$ the set of all points $x = (\xi_0, \cdots, \xi_{n-1})$ of Euclidean n-space with coordinates ξ_i $(i = 0, 1, \cdots, n-1)$, which admit the representation

$$\xi_i = \int_a^b w_i(t)\, d\sigma(t) \qquad (i = 0, 1, \cdots, n-1),$$

where $\sigma(t)$ is some non-decreasing function.

---

1) This theorem was first established in a somewhat less general form ($w_1(t) \equiv 1$) by F. Riesz [36b]. Later on the theorem turned up frequently being rediscovered by many authors. In particular, with the geometric proof given below it may be found in the articles of Schoenberg [38] and the author [11b] (who indicated an entire series of other predecessors). See also L. V. Kantorovič [6a].

The set $\mathfrak{R}$ has the following properties:

a) $\mathfrak{R}$ is a conical convex set, i.e., from $x, y \subset \mathfrak{R}$ and $\lambda \geq 0$ it follows that $\lambda x \subset \mathfrak{R}$, $x + y \subset \mathfrak{R}$.

b) $\mathfrak{R}$ contains the curve L whose equations are

$$X_0 = w_0(t), \cdots, X_{n-1} = w_{n-1}(t) \qquad (a \leq t \leq b).$$

c) $\mathfrak{R}$ lies entirely on one side of some hyperplane having one point in common with $\mathfrak{R}$, namely the origin of coordinates.

d) $\mathfrak{R}$ is a closed set.

Property a) is obvious.

Property b) follows because we may take as $\sigma(t)$ a piecewise constant function with a single jump equal to 1 at an arbitrary point t.

To prove property c) we recall that by the condition of the theorem there exists a positive combination

$$\omega(t) = \alpha_0^0 w_0(t) + \cdots + \alpha_{n-1}^0 w_{n-1}(t) > 0 \qquad (a \leq t \leq b),$$

and thus,

$$\alpha_0^0 \xi_0 + \alpha_1^0 \xi_1 + \cdots + \alpha_n^0 \xi_n = \int_a^b \omega(t)\, d\sigma(t) > 0,$$

if $x = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \subset \mathfrak{R}$ and $\sigma(t) \not\equiv \text{const}$, i.e., $x \neq 0$. Hence the hyperplane

$$\alpha_0^0 \xi_0 + \alpha_1^0 \xi_1 + \cdots + \alpha_{n-1}^0 \xi_{n-1} = 0$$

satisfies property c).

We now prove property d). Let

$$x^{(k)} = (\xi_0^{(k)}, \cdots, \xi_{n-1}^{(k)}) \subset \mathfrak{R} \qquad (k = 1, 2, \cdots),$$

i.e.,

$$\xi_j^{(k)} = \int_a^b w_j(t)\, d\sigma^{(k)}(t) \qquad (j = 0, 1, \cdots, n-1),$$

where $\sigma^{(k)}(t)$ are non-decreasing functions (which we also normalize in such a way that $\sigma^{(k)}(a) = 0$ $(k = 1, 2, \cdots)$); in addition, we let $x^{(k)} \to x = (\xi_0, \xi_1, \cdots, \xi_{n-1})$.

We shall show that $x \subset \mathfrak{R}$. Indeed, by setting

$$C = \sup\,(\alpha_0^0 \xi_0^{(k)} + \cdots + \alpha_{n-1}^0 \xi_{n-1}^{(k)}), \quad \mu = \min \omega(t) > 0,$$

we find that

$$0 \leq \sigma^{(k)}(t) \leq \sigma^{(k)}(b) \leq \frac{1}{\mu} \int_a^b \omega(t)\, d\sigma^{(k)}(t) \leq \frac{C}{\mu}.$$

Thus, by Helly's first theorem there exists a sub-sequence $\sigma^{(k_\nu)}(t)$ and a non-decreasing function $\sigma(t)$ such that at all points where $\sigma(t)$ is continuous it is true that

$$\lim_{\nu \to \infty} \sigma^{(k_\nu)}(t) = \sigma(t).$$

On the other hand, by Helly's second theorem it follows from

$$\xi_j^{(k_\nu)} = \int_a^b w_j(t)\, d\sigma^{(k_\nu)}(t) \qquad (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots)$$

that

$$\xi_j = \int_a^b w_j(t)\, d\sigma(t),$$

i.e., that $x = (\xi_0, \cdots, \xi_{n-1}) \subset \mathfrak{R}$. Therefore properties a), b), c), and d) are all proved.

Because $\mathfrak{R}$ is a conical set with its vertex at the origin of coordinates every supporting hyperplane for $\mathfrak{R}$ passes through 0.[1] Let

$$\alpha_0 \xi_0 + \cdots + \alpha_{n-1} \xi_{n-1} = 0 \tag{12}$$

be the equation of some supporting hyperplane for $\mathfrak{R}$; then the normalization of the coefficients α_j may be done in such a way that

$$\alpha_0 \xi_0 + \cdots + \alpha_{n-1} \xi_{n-1} \geq 0 \quad \text{for} \quad x = (\xi_0, \cdots, \xi_{n-1}) \subset \mathfrak{R}. \tag{13}$$

Since $L \subset \mathfrak{R}$ in this case also

$$\alpha_0 w_0(t) + \cdots + \alpha_{n-1} w_{n-1}(t) \geq 0 \qquad (a \leq t \leq b).$$

Conversely, the last inequality, as we showed in proving condition c), implies inequality (13), which means that the hyperplane (12) is supporting for $\mathfrak{R}$.[2]

By definition of the body $\mathfrak{R}$ the sequence $\{c_k\}_0^{n-1}$ admits a representation (11) if and only if $x_0 = (c_0, c_1, \cdots, c_{n-1}) \subset \mathfrak{R}$. Since $\mathfrak{R}$ is closed and convex, this will hold if and only if x_0 and $\mathfrak{R}$ lie on the same side of every supporting

1) The hyperplane H will be called *supporting* for the set $\mathfrak{M}$ if it is situated at zero distance from $\mathfrak{M}$ and if $\mathfrak{M}$ lies entirely on one side of H.

2) Thus, a hyperplane H is supporting for $\mathfrak{R}$ if and only if it passes through the origin of coordinates and the curve L lies on one side of it. Since $L \subset \mathfrak{R}$ it follows that $\mathfrak{R}$ is the smallest closed convex conical set (with vertex at the point O) containing the curve L.

hyperplane to $\mathfrak{R}$. [1] By what has been said about the hyperplanes (12) the last condition is equivalent to the condition of non-negativeness of the sequence $\{c_k\}_0^{n-1}$ relative to the sequence $\{w_k(t)\}_0^{n-1}$, since if $w_k(t)$ is real the sequence $\{c_k\}_0^{n-1}$ is non-negative definite if and only if all the c_k are real and the inequality

$$\alpha_0 w_0(t) + \cdots + \alpha_{n-1} w_{n-1}(t) \geq 0 \qquad (a \leq t \leq b)$$

implies the inequality

$$\alpha_0 c_0 + \cdots + \alpha_{n-1} c_{n-1} \geq 0.$$

Thus, our theorem is proved for the case of a finite sequence $\{w_k(t)\}$; with the help of Helly's theorem it is easy to extend it to the case of an infinite sequence, which we leave the reader to do.

The geometric method which we utilized in the second proof of the theorem also allows us to establish very simply the following proposition. [2]

Theorem 3. *If a finite sequence* $\{w_k(t)\}_0^{n-1}$ *of functions which are real and continuous in the interval* $<a, b>$ *has the property that for any real* α_k $(k = 0, 1, \cdots, n-1)$ *not all equal to zero, the function*

$$\alpha_0 w_0(t) + \cdots + \alpha_{n-1} w_{n-1}(t)$$

changes its sign in the interval $<a, b>$, *then for any sequence of real numbers* $c_0, c_1, \cdots, c_{n-1}$ *there is some non-decreasing function* $\sigma(t)$ *such that*

$$c_k = \int_a^b w_k(t)\, d\sigma(t) \qquad (k = 0, 1, \cdots, n-1).$$

Proof. We define the set $\mathfrak{R}$ as for the proof of Theorem 2. As before, $\mathfrak{R}$ has properties a) and b), i.e., there is a convex conical set which contains the curve L.

On the other hand, by the conditions of the theorem being proved, any hyperplane H passing through the origin of coordinates intersects the curve L and therefore $\mathfrak{R}$ (there are points of $\mathfrak{R}$ on both sides of H). Thus the origin O is an interior point of $\mathfrak{R}$ and therefore, since $\mathfrak{R}$ is a conical set with its vertex at O, $\mathfrak{R}$ is all of the Euclidean space. This proves the theorem.

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1) By a theorem of Minkowski: If a point x lies outside of a closed convex set $\mathfrak{R}$ then there exists a hyperplane for $\mathfrak{R}$ such that x and $\mathfrak{R}$ lie on different sides of it (cf. Theorem 5 of Article III).

2) Theorem 3 may also be found in the papers of Schoenberg [38] and of the author [11b]. Both Theorems 2 and 3 are very special consequences of general functional considerations presented in the latter article (see in this connection Article III).

4. **Applications of Theorem 2.** It is easy to see that Theorems 2, 1a–2a, and 7 of Chapter II of Article I are particular cases (with certain small additions) of Theorem 2.

As a new application of Theorem 1, we shall deduce another such proposition [1b, 11b].

a) *In order that there exist a non-decreasing function* $\sigma(t)$ $(-\tau \le t \le \tau)$, *which satisfies the equations*

$$c_k = \int_{-\tau}^{\tau} e^{kit} d\sigma(t) \qquad (k = 0, 1, 2, \cdots),$$

where $\{c_k\}$ *is a given finite or infinite sequence of complex numbers* $(c_0 > 0)$, *it is necessary and sufficient that the Toeplitz form*

$$\sum_{j,k=0}^{n} c_{j-k}\xi_j\bar{\xi}_k, \qquad \sum_{j,k=0}^{n-1} \{c_{j-k+1} - 2\cos\tau c_{j-k} + c_{j-k+1}\}\xi_j\bar{\xi}_k \tag{14}$$

be non-negative for any n *if the sequence* $\{c_k\}$ *is infinite, and for* $n = m$ *if it contains* $m + 1$ *members.*

Indeed, as can be shown, every trigonometric polynomial

$$T_n(t) = \sum_{k=-n}^{n} \alpha_k e^{kit} \qquad (\alpha_k = \bar{\alpha}_k;\ k = 0, 1, \cdots, n)$$

which is non-negative in the interval $(-\tau, \tau)$ admits the following representation:

$$T_n(t) = \left|\sum_{k=0}^{n} \xi_k e^{kit}\right|^2 + (\cos t - \cos\tau)\left|\sum_{0}^{n-1} \eta_k e^{kit}\right|^2 = \tag{15}$$

$$= \sum_{j,\,k=0}^{n} e^{(j-k)it}\xi_j\bar{\xi}_k + \frac{1}{2}\sum_{j,\,k=0}^{n-1}\left[e^{(j-k+1)it} - 2\cos\tau e^{(j-k)it} + e^{(j-k-1)it}\right]\xi_j\bar{\xi}_k.$$

In fact, if the polynomial $T_n(t)$ is non-negative in the interval $(-\tau, \tau)$ it has an even number of zeros $t_1 \le t_2 \le \cdots \le t_{2m}$ in the closed interval $\langle\tau, 2\pi - \tau\rangle$, and hence admits a representation

$$T_n(t) = G(t)\prod_{k=1}^{2m} \sin\frac{t_k - t}{2},$$

where $G(t)$ is a trigonometric polynomial of order $n - 2m$ which is everywhere non-negative.

On the other hand,

$$\sin\frac{t_k - t}{2} = A_k \sin\frac{\tau - t}{2} + B_k \sin\frac{\tau + t}{2}, \tag{*}$$

where

$$A_k = \frac{\sin\frac{\tau+t_k}{2}}{\sin\tau} \geq 0, \quad B_k = \frac{\sin\frac{t_k-\tau}{2}}{\sin\tau} \geq 0 \qquad (k = 1, 2, \cdots, m).$$

Using the relation

$$\sin\frac{\tau-\phi}{2}\sin\frac{\tau+\phi}{2} = \frac{1}{2}(\cos\tau - \cos\phi),$$

by multiplying the expressions (*) we find that

$$\prod_{k=1}^{2m} \sin\frac{t_k - t}{2} = R_1(t) + S_1(t)(\cos\tau - \cos t),$$

where $R_1(t)$ and $S_1(t)$ are certain non-negative polynomials of orders m and $m-1$, respectively.

Hence

$$T_n(t) = R(t) + (\cos\tau - \cos t)S(t),$$

where $R(t)$ and $S(t)$ are trigonometric polynomials, of orders n and $n-1$ respectively, which are everywhere non-negative. But this latter representation is equivalent to relation (15) if we apply the Fejér–Riesz theorem about the representation of everywhere non-negative trigonometric polynomials (cf. pp. 12–13, Article I).

Consequently, the sequence $\{c_k\}$ will be non-negative definite relative to the sequence of functions $\{e^{kit}\}$ in the interval $(-\tau, \tau)$ if for every polynomial $T_n(t)$ of the form of (15)

$$\mathfrak{C}\{T_n\} = \sum_{-n}^{n} a_k c_k = \sum_{j,k=0}^{n} c_{j-k}\xi_j\overline{\xi}_k + \sum_{j,k=0}^{n-1}\{c_{j-k+1} - 2\cos\tau\, c_{j-k} + c_{j-k+1}\}\eta_j\overline{\eta}_k \geq 0.$$

Because the numbers ξ_j and η_j in this inequality are completely independent, it is equivalent to the condition of non-negativeness of the two forms (14).

We shall prove another such proposition [8, 11b].

b) *There corresponds to two systems of complex numbers* $\{w_k\}$ *and* $\{z_k\}$ $(\Im z_k > 0;\ k = 1, 2, \cdots)$ *a non-decreasing function* $\sigma(t)$ $(a \leq t \leq b)$ *such that*

$$w_k = \int_a^b \frac{d\sigma(t)}{t - z_k} \qquad (k = 1, 2, \cdots, n),$$

if and only if the Hermitian forms

$$\sum_{j,k=1}^{n} \frac{w_j(z_j - a) - \overline{w}_k(\overline{z}_k - a)}{z_j - \overline{z}_k}\xi_j\overline{\xi_k}, \quad \sum_{j,k=1}^{n} \frac{w_j(b - z_j) - \overline{w}_k(b - \overline{z}_k)}{z_j - \overline{z}_k}\xi_j\overline{\xi_k}$$

are non-negative for any n if the sequence $\{z_k\}$ (and consequently $\{w_k\}$ also) is infinite, and for $n = m$ if the sequence $\{z_k\}$ has only m members.

Indeed, by using a theorem of Lukacs, [1] every non-negative function

$$g(t) = \sum_1^n \frac{\alpha_k}{t-z_k} + \sum_1^n \frac{\overline{\alpha_k}}{t-\overline{z}_k} \geq 0 \qquad (a \leq t \leq b)$$

may be represented in the following form:

$$g(t) = \frac{\sum_0^{2n-1} A_k t^k}{\left|\prod_1^n (t-z_k)\right|^2} = (t-a)\left|\frac{\sum_0^{n-1} \xi_k' t^k}{\prod_1^n (t-z_k)}\right|^2 + (b-t)\left|\frac{\sum_0^{n-1} \eta_k' t^k}{\prod_1^n (t-z_k)}\right|^2 =$$

$$= (t-a)\left|\sum_1^n \frac{\xi_k}{t-z_k}\right|^2 + (b-t)\left|\sum_1^n \frac{\eta_k}{t-z_k}\right|^2,$$

or using the identities

$$\frac{t-a}{(t-z_k)(t-\overline{z}_l)} = \frac{1}{z_k-\overline{z}_l}\left\{\frac{z_k-a}{t-z_k} - \frac{\overline{z}_l-a}{t-\overline{z}_l}\right\},$$

$$\frac{b-t}{(t-z_k)(t-\overline{z}_l)} = \frac{1}{z_k-\overline{z}_l}\left\{\frac{b-z_k}{t-z_k} - \frac{b-\overline{z}_l}{t-\overline{z}_l}\right\}$$

in the form

$$g(t) = \sum_{k,l=1}^n \frac{1}{z_k-z_l}\left\{\frac{z_k-a}{t-z_k} - \frac{\overline{z}_l-a}{t-\overline{z}_l}\right\}\xi_k\,\xi_l + \sum_{k,l=1}^n \frac{1}{z_k-\overline{z}_l}\left\{\frac{b-\overline{z}_k}{t-z_k} - \frac{b-\overline{z}_l}{t-\overline{z}_l}\right\}\eta_k\,\overline{\eta}_l.$$

Therefore, the sequence $\{w_k\}_1^n$ is non-negative definite relative to the sequence $\left\{\frac{1}{t-z_k}\right\}^n$ if

$$\sum_1^n \alpha_k w_k + \sum_1^n \overline{\alpha}_k \overline{w}_k = \sum_{k,l=1}^n \frac{w_k(z_k-a) - \overline{w}_l(\overline{z}_l-a)}{z_k-\overline{z}_l}\,\xi_k\overline{\xi}_l +$$

$$+ \sum_{k,l=1}^n \frac{w_k(b-z_k) - \overline{w}_l(b-\overline{z}_l)}{z_k-\overline{z}_l}\,\eta_k\overline{\eta}_l \geq 0$$

for any ξ_k and η_k $(k = 1, 2, \cdots, n)$, which proves proposition b).

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1) Cf. page 35.

§2

Now let I denote a finite or infinite (on one or both sides) closed interval on the real axis. As before, we denote by E a certain collection of complex functions defined on I and denote by $\breve{E}$ the real linear envelope of the real and imaginary parts of the functions of E.

We shall agree to call a function $\phi(t) \subset E$ *normal* if it satisfies the following two conditions:

1°. *The function $\phi(t)$ is continuous in I everywhere except at a finite number of points $a_1, a_2, \cdots, a_n$ (these points vary in general with ϕ).*

2°. *There corresponds to the function ϕ at least one non-negative function $\omega_\phi(t) \subset \breve{E}$ for which*

$$\lim_{t \to a_i} \frac{\phi(t)}{\omega_\phi(t)} = 0 \qquad (i = 1, 2, \cdots, m), \tag{16}$$

and also, if I is an infinite interval,

$$\lim_{t \to \infty} \frac{\phi(t)}{\omega_\phi(t)} = 0 \qquad (t \subset I). \tag{17}$$

Let $\phi(t)$ $(t \subset I)$ be a function which satisfies the condition 1° and $\sigma(t)$ $(t \subset I)$ be a non-decreasing function. Consider the integral

$$\int_{I'} \phi(t)\, d\sigma(t), \tag{18}$$

where I' is obtained from I by removal of arbitrary neighborhoods of the points $a_1, \cdots, a_m, \infty$. If, when we contract these neighborhoods to zero arbitrarily the integral (18) approaches some limit, we shall denote this limit by

$$\overset{*}{\int_I} \phi(t)\, d\sigma(t).$$

We shall suppress the asterisk if at every point a where the function $\phi(t)$ is discontinuous either $\sigma(a + 0) - \sigma(a - 0) = 0$ or $\liminf_{t \to a} |\phi(t)| = 0$.

We may speak of the integral

$$\int_I \phi(t)\, d\sigma(t)$$

as of an *improper* integral.

For conciseness we introduce the following terms: The function $\omega(t)$ $(t \subset I)$ will be termed *P-positive* if in every closed finite part of the interval the greatest

lower bound of the values of $\omega(t)$ is greater than zero. It is obvious that every positive continuous function in I is P-positive.

By enlarging somewhat upon the method by which Theorem 1 was proved we may immediately establish a more general proposition:

Theorem 4. *If the collection $\breve{E}$ contains at least one P-positive function $\omega_0(t)$, then there corresponds to each non-negative definite functional $\mathfrak{P}(\phi)$ on E at least one non-decreasing function $\sigma(t)$ $(t \subset I)$ such that for every normal function $\phi \subset E$*

$$\mathfrak{P}(\phi) = \int_I \phi(t)\, d\sigma(t). \tag{19}$$

Proof. Just as for the proof of the second part of Theorem 1, we select an arbitrary point a within I, define the function $\xi_\tau(t)$ by equalities (8) and (9), form the sequence $\{t_k\}_1^\infty$ which is dense in I and contains the endpoints of I (if there be such), extend the functional $\mathfrak{P}$ on the set $E + \{\xi_{t_k}(t)\}_1^\infty$ so as to preserve positiveness, and construct the non-decreasing function $\sigma(t)$ $(t \subset I)$ by formulae (10) and (10′).

We shall show that the function $\sigma(t)$ thus obtained satisfies condition (19).

We note at the start that the function $\sigma(t)$ has the following property. Let x be a point of the interval I and let there be at least one non-negative function $\omega(t) \subset \breve{E}$ which satisfies the condition:

$$\lim_{t \to x} \omega(t) = \infty; \tag{20}$$

then

$$\sigma(x + 0) - \sigma(x - 0) = 0.$$

In fact, if (20) holds, for any $\epsilon > 0$ we may choose points t' and $t'' \subset \{t_k\}$ such that

$$t' < x < t'', \quad \xi''_{t''}(t) - \xi_{t'}(t) < \epsilon\omega(t);$$

but then

$$\sigma(t'') - \sigma(t') = \mathfrak{P}(\xi_{t''}) - \mathfrak{P}(\xi_{t'}) < \epsilon\mathfrak{P}(\omega),$$

which proves our assertion.

In particular, it follows from this that if $\phi(t)$ is a normal function with a as a point of discontinuity and

$$\liminf_{t \to a} |\phi(t)| > 0, \tag{21}$$

then

$$\sigma(a + 0) - \sigma(a - 0) = 0;$$

in this case in fact, using (16), $\lim_{t \to a} \omega_\phi(t) = \infty$.

Since the real and imaginary parts of every normal function are normal in $\breve{E}$, it is sufficient for the proof of (19) to consider only the case when $\phi(t) \subset \breve{E}$. Moreover, since the points $\pm \infty$ (in the case where the interval I is infinite) play the same role as the singular points a_i $(i = 1, 2, \cdots, m)$,[1] we shall carry out the proof under the assumption that $\phi(t)$ is continuous in the interval I, which we will assume to be infinite.

Therefore, let the function $\phi(t) \subset E$ continuous in the infinite interval I be normal. Since this function is normal, there corresponds to it some non-negative function $\omega_\phi(t) \subset \breve{E}$ such that for any $\epsilon > 0$

$$-\epsilon\omega_\phi(t) < \phi(t) < \epsilon\omega_\phi(t) \quad \text{for} \quad t \subset I, \ |t| > N_\epsilon, \tag{22}$$

where N_ϵ is a positive number dependent upon ϵ.

We shall denote by I_{N_ϵ} a finite interval whose end points are members of the sequence $\{t_k\}_1^\infty$ and which contains the intersection of the interval I with the interval $<-N_\epsilon, N_\epsilon>$. For any $\epsilon' > 0$ we may select in the interval $I_{N_\epsilon} = <\alpha, \beta>$ points $\alpha = s_1 < s_2 < \cdots < s_n = \beta$ belonging to the sequence $\{t_k\}_1^\infty$, such that

$$-\epsilon'\omega_0(t) < \phi(t) - \sum_1^{n-1} \phi(s_k)[\xi_{s_{k+1}}(t) - \xi_{s_k}(t)] < \epsilon'\omega_0(t) \quad \text{for} \quad t \subset I_{N_\epsilon}. \tag{23}$$

Then by (22) for any $t \subset I$

$$-\epsilon\omega_\phi(t) - \epsilon'\omega_0(t) < \phi(t) - \sum_1^{n-1} \phi(s_k)[\xi_{s_{k+1}}(t) - \xi_{s_k}(t)] < \epsilon\omega_\phi(t) + \epsilon'\omega_0(t).$$

Therefore, by the non-negativeness of the functional $\mathfrak{P}$

$$-\epsilon\mathfrak{P}(\omega_\phi) - \epsilon'\mathfrak{P}(\omega_0) < \mathfrak{P}(\phi) - \sum_1^{n-1} \phi(s_k)[\sigma(s_{k+1}) - \sigma(s_k)] < \epsilon\mathfrak{P}(\omega_\phi) + \epsilon'\mathfrak{P}(\omega_\phi). \tag{24}$$

Taking the limit as $\max(s_{k+1} - s_k) \to 0$ we obtain

$$-\epsilon\mathfrak{P}(\omega_\phi) - \epsilon'\mathfrak{P}(\omega_0) < \mathfrak{P}(\phi) - \int_{I_{N_\epsilon}} \phi(t)\,d\sigma(t) < \mathfrak{P}(\omega_\phi) + \epsilon'\mathfrak{P}(\omega_0). \tag{25}$$

Since ϵ' is an arbitrary positive quantity, we therefore have

---

1) That is, if we do not consider it necessary to give any further explanation of the fact that for the singular points a inequality (21) implies the equality $\sigma(a+0) - \sigma(a-0) = 0$, which we have done above.

$$-\epsilon \mathfrak{P}(\omega_\phi) < \mathfrak{P}(\phi) - \int_{I_{N_\epsilon}} \phi(t)\, d\sigma(t) \leq \epsilon \mathfrak{P}(\omega_\phi), \tag{26}$$

which proves equality (19).

Applications of Theorem 4. a) Let $E = \{t^k\}_0^\infty$ $(t \subset I)$, where I is infinite or half-infinite. Since

$$\lim_{|t| \to \infty} \frac{t^k}{t^{2m}} = 0 \quad \text{for} \quad k < 2m,$$

all members of the sequence E are normal functions. Therefore, by Theorem 4, the real sequence of real numbers $\{s_k\}_0^\infty$ admits a representation

$$s_k = \int_I t^k d\sigma(t) \qquad (k = 0, 1, 2, \cdots), \tag{27}$$

if and only if the sequence $\{s_k\}_0^\infty$ is non-negative definite in the interval I.

Taking $I = (-\infty, \infty)$ gives a criterion for problem (27) to be solvable, due to Hamburger. In fact as was indicated earlier,[1] the sequence $\{s_k\}_0^\infty$ is non-negative definite in the interval $(-\infty, \infty)$ if and only if the quadratic form

$$\sum_{i,j=0}^{n} s_{i+j} \xi_i \xi_j$$

is non-negative for any $n = 0, 1, 2, \cdots$.

Taking $I = (0, \infty)$ we come to a criterion for the corresponding problem (27) to be solvable, indicated by Stieltjes, which is comprised in the fact that both of the forms

$$\sum_{i,j=0}^{n} s_{i+j} \xi_i \xi_j, \quad \sum_{i,j=0}^{n} s_{i+j+1} \xi_i \xi_j \tag{28}$$

are non-negative for any $n = 0, 1, 2, \cdots$. In order to obtain the latter criterion it is sufficient to note that if some polynomial

$$P(t) = \sum_0^n A_k t^k$$

is non-negative for $0 \leq t \leq \infty$, then it admits a representation

$$P(t) = \left(\sum_0^{\left[\frac{m}{2}\right]} \xi_k t^k \right)^2 + t \left(\sum_0^{\left[\frac{m-1}{2}\right]} \eta_k t^k \right)^2,$$

1) Cf. Article I, Chapter I, Theorem 1.

where ξ and η are real numbers, which means the sequence $\{s_k\}_0^\infty$ is non-negative in the interval $(0, \infty)$ if and only if the forms (28) are non-negative.

b) Let $R_{a,b}$ denote the set of all rational numbers lying in the finite or infinite interval (a, b). Let $E = \{e^{xt}\}$ $(-\infty < t < \infty)$, where x runs through the set $R_{a,b}$. The collection E consists exclusively of normal functions. In fact, if $\phi(t) = e^{x_0 t}$, where $x_0 \subset R_{a,b}$, then taking numbers $x' < x_0$ and $x'' > x_0$ in $R_{a,b}$ we may set

$$\omega_\phi(t) = e^{x't} + e^{x''t} \qquad (-\infty < t < \infty).$$

In this manner, by Theorem 4 to every positive definite functional $\mathfrak{F}$ in E there corresponds a non-decreasing function $\sigma(t)$ $(-\infty < t < \infty)$ such that

$$f(x) = \mathfrak{F}(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} d\sigma(t) \qquad (x \subset R_{a,b}).$$

We shall now specify what kind of function $f(x)$ must be in order for $\mathfrak{F}$ to be non-negative definite in E. The condition that $\mathfrak{F}$ is non-negative definite is comprised in the given case in the fact that from the inequality

$$\phi(t) = \sum_1^n c_k e^{x_k t} \geq 0 \quad \text{for} \quad -\infty < t < \infty \qquad (x_k \subset R_{a,b};\ k = 1, 2, \cdots, n) \quad (29)$$

must follow the inequality

$$\mathfrak{F}(\phi) = \sum_1^n c_k f(x_k) \geq 0.$$

Assuming $x_1 < x_2 < \cdots < x_n$ we reduce the rational numbers x_k to a common denominator; let

$$x_k = \frac{m_k}{N} \qquad (k = 1, 2, \cdots, n),$$

where N and m_k $(k = 1, 2, \cdots, n)$ are whole numbers. Let

$$u = e^{\frac{1}{N}t};$$

then inequality (29) may be written in the following form

$$\phi(t) = P(u) = \sum_1^n c_k u^{m_k} \geq 0 \quad \text{for} \quad 0 < u < \infty,$$

or in the form

$$P_1(u) = u^{-m_1} P(u) = \sum_1^n c_k u^{m_k - m_1} \geq 0 \quad \text{for} \quad 0 < u < \infty.$$

But the polynomial $P_1(u)$, which is non-negative in the interval $(0, \infty)$, admits the representation

$$P_1(u) = \left(\sum_0^p \xi_k u^k\right)^2 + u\left(\sum_0^q \eta_k u^k\right)^2 = \sum_{j,k=0}^{p} u^{j+k}\xi_j\xi_k +$$

$$+ \sum_0^q u^{j+k+1}\eta_j\eta_k, \qquad \left(p = \left[\frac{m_n - m_1}{2}\right],\ q = \left[\frac{m_n - m_1 - 1}{2}\right]\right);$$

consequently

$$\varphi(t) = \sum_{j,k=0}^{p} e^{\frac{j+k+m_1}{N}t}\xi_j\xi_k + \sum_{j,k=0}^{q} e^{\frac{j+k+m_1+1}{N}}\eta_j\eta_k.$$

Hence inequality (29) is equivalent to the following:

$$\mathfrak{B}(\varphi) = \sum_{j,k=0}^{p} f\left(\frac{j+k+m_1}{N}\right)\xi_j\xi + \sum_{j,k=0}^{q} f\left(\frac{j+k+m_1+1}{N}\right)\eta_j\eta_k \geqslant 0. \qquad (30)$$

On the other hand, the last inequality will be fulfilled if for any $x_j \subset R_{a,b}$ $(j = 1, 2, \cdots, n)$ such that $x_j + x_k \subset R_{a,b}$ $(j, k = 1, 2, \cdots, n)$ the form

$$\Phi = \Sigma f(x_j + x_k)\xi_j\xi_k \geq 0, \qquad (31)$$

since every one of the forms which figure in (30) may be represented as some form Φ. Thus condition (31) is sufficient for the functional $\mathfrak{B}$ to be non-negative definite. Since

$$\Phi = \mathfrak{B}\left\{\left(\sum_1^n \xi_j e^{x_j t}\right)^2\right\},$$

it is also necessary. We arrive at the proposition:

For a function $f(x)$ defined in $R_{a,b}$ to admit there a representation

$$f(x) = \int_{-\infty}^{\infty} e^{xt}\,d\sigma(t), \qquad (32)$$

where $\sigma(t)$ $(-\infty < t < \infty)$ is some non-decreasing function, it is necessary and sufficient that for any $x_j \subset R_{a,b}$ $(j = 1, 2, \cdots, n)$ such that $a < x_j + x_k < b$ $(j, k = 1, 2, \cdots, n)$ the form

$$\sum_{j,k=1}^{n} f(x_j + x_k)\xi_j\xi_k \qquad (33)$$

is non-negative.

From this proposition there readily follows a theorem of Widder [42] which says that a function $f(x)$ defined within the interval (a, b) admits a representation (32) if and only if it is continuous and for any x_j $(j = 1, 2, \cdots, n)$ which together with

$x_j + x_k$ $(j, k = 1, 2, \cdots, n)$ lies within (a, b) the forms (33) are non-negative.

§4

Besides normal functions it is also appropriate to consider semi-normal functions which we define in the following way.

The function $\phi(t) \subset \breve{E}$ will be called *semi-normal* if it satisfies condition 1° (cf. p. 136) and if to it there corresponds at least one function $\omega_\phi(t) \subset \breve{E}$ which is non-negative and has a finite number of points of discontinuity [1] for which [instead of (16) and (17)] there hold the relations

$$\liminf_{t \to a_i} \frac{\phi(t)}{\omega_\phi(t)} \geq 0 \qquad (i = 1, 2, \cdots, m) \tag{33}$$

and

$$\liminf_{|t| \to \infty} \frac{\phi(t)}{\omega_\phi(t)} \geq 0 \qquad (t \subset I). \tag{34}$$

Obviously, every non-negative function $\phi(t) \subset \breve{E}$ which has a finite number of points of discontinuity is semi-normal (ω_ϕ may be taken equal to $\phi(t)$).

Theorem 5. *The function $\sigma(t)$ which was constructed in the proof of Theorem 4 still has the additional property that for any semi-normal function $\phi(t) \subset \breve{E}$*

$$\mathfrak{P}(\phi) \geq \overset{*}{\int_I} \phi(t)\, d\sigma(t). \tag{35}$$

Proof. As in the proof of Theorem 4 the interval I may be assumed infinite and the function $\phi(t)$ continuous in I.

By (34) we have for any $\epsilon > 0$ the inequality

$$-\epsilon\omega_\phi(t) < \phi(t) \quad \text{for} \quad t \subset I, \quad |t| > N_\epsilon,$$

where $N_\epsilon > 0$ is a number depending on ϵ. Upon comparing this with relation (22) we easily verify that all of the reasoning following after (22) remains valid if we keep only the left inequalities of relations (23), (24), (25), and (26). Therefore

$$\int_{I_{N_\epsilon}} \phi(t)\, d\sigma(t) \leq \mathfrak{P}(\phi) + \epsilon\mathfrak{P}(\omega_\phi). \tag{36}$$

Hence

---

1) The condition that the function $\omega_\phi(t)$ has a finite number of points of discontinuity is probably superfluous (it did not come into the definition of normal functions) but we do not know how to get rid of it.

$$\int_I \phi(t)\,d\sigma(t) \leq \mathfrak{P}(\phi),$$

if the integral on the left side exists. If $\phi(t) \geq 0$, then this integral by (36) certainly exists. In this way we may consider (35) proved for non-negative semi-normal functions (the asterisk appears for the integral if $\phi(t)$ has discontinuities in I). In particular, the integral

$$\overset{*}{\int_I} \omega_\phi(t)\,d\sigma(t) \leq \mathfrak{P}(\omega_\phi)$$

exists. Furthermore, since for a definite choice of points $\alpha, \beta \subset I$ $(\alpha < \beta)$ and a number $M > 0$ the function

$$\psi(t) = \phi(t) + \omega_\phi(t) + M[\xi_\beta(t) - \xi_\alpha(t)] \qquad (t \subset I)$$

is non-negative and hence the integral

$$\overset{*}{\int_I} \psi(t)\,d\sigma(t)$$

exists, the integral

$$\overset{*}{\int_I} \phi(t)\,d\sigma(t)$$

also exists, thus proving the theorem.

We note the following consequence of Theorems 1, 4, and 5.

Theorem 6. *Let the real functions $u_k(t)$ $(k = 0, 1, \cdots, n)$ be defined and continuous on the infinite closed interval I and let*

$$u_n(t) > 0, \quad \lim_{|t| \to \infty} \frac{u_k(t)}{u_n(t)} = 0 \qquad (k = 0, 1, \cdots, n-1,\ t \subset I).$$

In order that to the sequence of numbers $\{c_k\}_0^n$ there should correspond a non-decreasing function $\sigma(t)$ $(t \subset I)$ and a non-negative constant M such that

$$\begin{cases} c_k = \int_I u_k(t)\,d\sigma(t) & (k = 0, 1, \cdots, n-1), \\ c_n = \int_I u_n(t)\,d\sigma(t) + M, \end{cases} \tag{37}$$

it is necessary and sufficient that the sequence $\{c_k\}_0^n$ be non-negative definite relative to the sequence $\{u_k(t)\}_0^n$.

Proof. The necessity of the indicated condition follows from the inequality

$$\sum_0^n A_k u_k(t) \geq 0$$

which implies the inequality

$$A_n = \lim_{|t| \to \infty} \sum_0^n \frac{A_k u_k(t)}{u_n(t)} \geq 0,$$

and therefore by (37) the inequality

$$\sum_0^n A_k c_k = \int_J \left(\sum_0^n A_k u_k(t) \right) d\sigma(t) + M A_n \geq 0.$$

Conversely, if the sequence $\{c_k\}_0^n$ is non-negative definite, then the functional $\mathfrak{F}$ defined on $E = \{u_k(t)\}_0^n$ by the equalities

$$\mathfrak{F}(u_k) = c_k \qquad (k = 0, 1, \cdots, n)$$

is positive definite in E. Because the functions $u_k(t)$ $(k = 0, 1, \cdots, n-1)$ are normal in E and $u_n(t) > 0$ $(t \subset I)$ the existence of representation (37) follows from Theorem 4.

From this theorem follows the theorem of Fischer (Theorem 5, Chapter I, Article I), if we set $u_k(t) = t^k$ $(k = 0, 1, \cdots, 2m-2)$, $I = (-\infty, \infty)$. The following proposition serves as an interesting supplement to Theorem 6.

Theorem 7. *Let the real functions $u_k(t)$ $(k = 0, 1, \cdots, n-1)$ be defined and continuous on the infinite closed interval I.*

If the sequence $\{c_k\}_0^{n-1}$ is positive definite in I (relative to the sequence of functions $\{u_k(t)\}_0^{n-1}$), then for sufficiently large N the sequence $\{c_k\}_0^{n-1}$ is also positive definite in I_N, the intersection of the interval I with the interval $(-N, N)$. Therefore, if there exists at least one positive linear combination $\sum_0^{n-1} A_k u_k(t) > 0$ for $t \subset I$, then the sequence $\{c_k\}_0^{n-1}$, which is positive definite in I, admits a representation

$$c_k = \int_{I_N} u_k(t)\, d\sigma(t) \qquad (k = 0, 1, \cdots, n-1),$$

where $\sigma(t)$ $(t \subset I_N)$ is some non-decreasing function.

Proof. Assume to the contrary that we can construct a sequence of numbers

$$N_1 < N_2 < N_3 < \cdots < N_m \to \infty$$

and a sequence of polynomials

$$P^{(m)}(t) = \sum_{0}^{n-1} A_k u_k(t)$$

such that

$$P^{(m)}(t) \geq 0 \quad (u \not\equiv 0) \quad \text{for} \quad t \subset I_{N_m},$$

and so that

$$\mathfrak{P}(P^{(m)}) \leq 0 \qquad (m = 1, 2, \cdots). \tag{38}$$

Without loss of generality we may consider the functions $u_k(t)$ $(k = 0, 1, \cdots, n)$ to be linearly independent in I (otherwise we would exclude dependent functions from consideration) and also that

$$\sum_{k=0}^{n} |A_k^{(m)}| = 1 \qquad (m = 1, 2, \cdots).$$

We may therefore select a sequence $k_1 < k_2 < k_3 < \cdots$ such that

$$\lim_{m \to \infty} A_j^{(k_m)} = A.$$

In this manner we come to the polynomial

$$P(t) = \sum_{0}^{n} A_k u_k(t),$$

which obviously is non-negative on I and not identically equal to zero, since

$$\sum_{0}^{n} |A_k| = 1.$$

Therefore we must have

$$\mathfrak{P}(P) > 0,$$

while it follows from (38) that

$$\mathfrak{P}(P) \leq 0.$$

We have reached a contradiction. The first assertion of the theorem is established. The second assertion follows directly from the first by Theorem 2.

§5

1. In this paragraph we shall deduce several applications of Theorems 4 and 5 in the theory of N-functions [1] (cf. Chapter II of Article I). These applications

1) In distinction with Chapter II of Article I we shall now mean by an N-function any function $F(z)$ which is regular within the upper half-plane and has a non-negative imaginary part there.

are based on the following remarks, which immediately follow from Theorems 4, 5.

Let the collection $\breve{E}$ *contain* $n+1$ *functions* $\omega_0(t), \omega_1(t), \cdots, \omega_n(t)$ *such that every function* $\phi \subset E$ *admits a representation*

$$\phi(t) = \sum_0^n c_k(\phi)\,\omega_k(t) + \phi^*(t), \tag{39}$$

where $c_k(\phi)$ $(k = 0, 1, \cdots, n)$ *are complex numbers and* $\phi^*(t)$ *is a normal or semi-normal function; moreover, let the function* $\omega_0(t)$ *be continuous and semi-normal (for example, non-negative).*

Under these conditions there corresponds to every functional $\mathfrak{P}$, *non-negative in* E, *a system of numbers* $a_0 \geq 0$, $a_1, \cdots, a_n$ *and a non-decreasing function* $\sigma(t)$ *such that*

$$\mathfrak{P}(\phi) = \sum_0^n a_k c_k(\phi) + \int_I \{\phi(t) - \sum_1^n c_k(\phi)\,\omega_k(t)\}\,d\sigma(t) \tag{40}$$

or

$$\mathfrak{P}(\phi) \geq \sum_0^n a_k c_k(\phi) + \int_I \{\phi(t) - \sum_1^n c_k(\phi)\,\omega_k(t)\}\,d\sigma(t), \tag{41}$$

depending upon whether the function $\phi^*(t)$ *is normal or semi-normal.* [1]

In fact, according to Theorems 4 and 5 there exist a non-decreasing function $\sigma(t)$ $(t \subset I)$ and a constant $a_0 \geq 0$ such that

$$\begin{aligned} \mathfrak{P}(\omega_0) &= \int_I \omega_0(t)\,d\sigma(t) + a_0, \\ \mathfrak{P}(\phi^*) &= \int_I \phi^*(t)\,d\sigma(t), \end{aligned} \tag{42}$$

whenever the function $\phi^*(t)$ is normal. But then from (39)

$$\mathfrak{P}(\phi) = \sum_1^n c_k(\phi)\,\mathfrak{P}(\omega_k) + \mathfrak{P}\{c_0(\phi)\,\omega_0(t) + \phi^*(t)\} = \sum_0^n a_k c_k(\phi) + \int_I \{c_0(\phi)\,\omega_0(t) + \phi^*(t)\}\,d\sigma(t),$$

where

$$a_k = \mathfrak{P}(\omega_k) \qquad (k = 0, 1, \cdots, n).$$

Since

$$c_0(\phi)\,\omega_0(t) + \phi^*(t) = \phi(t) - \sum_1^n c_k(\phi)\,\omega_k(t),$$

equality (40) is proved.

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1) For the formulation of this proposition we did not strive for the greatest possible generality, but had only the later applications in mind.

If a function $\phi^*(t)$ is semi-normal the = sign must be replaced by $\geq$ and an asterisk must be attached to the integral. By reasoning analogous to the previous we obtain inequality (41). In this way our remark is proved.

2. Let there be given a one-to-one correspondence between the elements α of a certain finite, denumerable or non-denumerable set $\mathfrak{M}$ and the fractions

$$\frac{1}{(t-z_\alpha)^{k_\alpha}} \quad (-\infty < t < \infty), \tag{43}$$

where z_α is a point in the upper half-plane $\Im z > 0$ and k_α is a natural number.

We shall determine the general form of a non-negative functional on the set E of all functions (43) for the case when at least one $k_\alpha = 1$. For example, let $k_1 = 1$. If $z_1 = x_1 + iy_1$ then the functions

$$u_1(t) = \frac{t-x_1}{(t-x_1)^2+y_1^2} = \Re\left\{\frac{1}{t-z_1}\right\},$$

$$v_1(t) = \frac{y_1}{(t-x_1)^2+y_1^2} = \Im\left\{\frac{1}{t-z_1}\right\}$$

as well as their real linear combinations belong to $\breve{E}$. We introduce the functions

$$\omega_0(t) = \frac{1}{t^2+1}, \quad \omega_1(t) = \frac{t}{t^2+1}.$$

If they are not contained in $\breve{E}$ we define an extension of the functional $\mathfrak{F}$ to them such that it remains non-negative on the set $F = \Gamma_1 + \{\omega_0, \omega_1\}$. According to the lemma, §1, this is possible, since for large enough N

$$-Nv_1(t) < \omega_0(t) < Nv_1(t)$$

$$u_1(t) - Nv_1(t) < \omega_1(t) < u_1(t) + Nv_1(t).$$

The set F satisfies the conditions of Theorem 6; in fact, $\omega_0(t) > 0$ and

$$\frac{1}{(t-z_\alpha)^{k_\alpha}} = \begin{cases} z_\alpha\omega_0(t) + \omega_1(t) + g_\alpha(t) & \text{for } k_\alpha = 1 \\ \omega_0(t) + g_\alpha(t) & \text{for } k_\alpha = 2 \\ g_\alpha(t) & \text{for } k_\alpha > 2, \end{cases}$$

where $g_\alpha(t)$ is a function which satisfies the condition

$$\lim_{|t|\to\infty} \frac{g_\alpha(t)}{\omega_0(t)} = 0.$$

Therefore, by Theorem 6, there exist real constants $a_0 > 0$, a_1 and a non-decreasing function $\sigma(t)$ $(-\infty < t < \infty)$ such that

$$\mathfrak{P}\left\{\frac{1}{(t-z_\alpha)^{k_\alpha}}\right\} = \begin{cases} a_0 z_\alpha + a_1 + \int_{-\infty}^{\infty}\left[\frac{1}{t-z_\alpha} - \frac{t}{1+t^2}\right] d\sigma(t) & \text{for } k_\alpha = 1 \\ a_0 + \int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-z_\alpha)^2} & \text{for } k_\alpha = 2 \\ \int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-z_\alpha)^{k_\alpha}} & \text{for } k_\alpha > 2. \end{cases} \tag{44}$$

We introduce the function of a complex variable

$$F(z) = a_0 z + a_1 + \int_{-\infty}^{\infty}\left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] d\sigma(t) =$$

$$= a_0 z + a_1 + \int_{-\infty}^{\infty} \frac{1+zt}{t-z}\, d\sigma_1(t) \quad \left(\sigma_1(t) = \int_c^t (1+t^2)\, d\sigma(t)\right). \tag{45}$$

In accordance with a theorem of Chapter II of Article I this function is regular in the upper half-plane and maps it onto part of itself ($F(z)$ is an N-function) and conversely, if the function $F(z)$ has these properties, then it admits a representation (45) in which $a_0 \geq 0$, $a_1 \gtreqless 0$ and $\sigma(t)$ is a non-decreasing function. With the help of the function $F(z)$ relations (44) may be written in a very simple form:

$$\mathfrak{P}\left\{\frac{1}{(t-z_\alpha)^{k_\alpha}}\right\} = \frac{F^{(k_\alpha - 1)}(z_\alpha)}{(k_\alpha - 1)!} \qquad (\alpha \subset \mathfrak{M}). \tag{46}$$

We shall now show that, conversely, if a functional $\mathfrak{P}$ is defined on the set of functions (43) with the help of the function $F(z)$ by the equations (46) where $F(z)$ is some N-function, then it is non-negative definite on E.

Indeed, if

$$g(t) = \sum_1^n \frac{\gamma_j}{(t-z_{\alpha_j})^{k_{\alpha_j}}} + \sum_1^n \frac{\bar{\gamma}_j}{(t-\bar{z}_{\alpha_j})^{k_{\alpha_j}}} \geq 0 \quad (-\infty < t < \infty),$$

then in the expansion about the point at infinity

$$g(t) = \frac{C_0}{t} + \frac{C_1}{t^2} + \cdots$$

the coefficients $C_0 = 0$, $C_1 \geq 0$. Therefore, by (45) and (46),

$$\mathfrak{P}\{g\} = \sum_1^n \gamma_j \mathfrak{P}\left\{\frac{1}{(t-z_{\alpha_j})^{k_{\alpha_j}}}\right\} + \sum_1^n \bar{\gamma}_j \mathfrak{P}\left\{\frac{1}{(t-\bar{z}_{\alpha_j})^{k_{\alpha_j}}}\right\} =$$

$$= a_0 C_1 + a_1 C_0 + \int_{-\infty}^{\infty} \left[g(t) - C_0 \frac{t}{1+t^2}\right] d\sigma(t) = a_0 C_1 + \int_{-\infty}^{\infty} g(t)\, d\sigma(t) \geq 0, \quad (47)$$

which proves our assertion. We have arrived in this way at the following proposition.

A. *For the existence of an N-function $F(z)$ satisfying the equations*

$$\frac{F^{(k_\alpha - 1)}(z_\alpha)}{(k_\alpha - 1)!} = \omega_\alpha \quad (\Im z_\alpha > 0;\ \alpha \subset \mathfrak{M}),$$

where at least one of the numbers k_α equals 1, it is necessary and sufficient that the functional $\mathfrak{P}$ defined on the set of functions $E = \left\{\frac{1}{(t-z_\alpha)^{k_\alpha}}\right\}$ $(\alpha \subset \mathfrak{M})$ by the equalities

$$\mathfrak{P}\left\{\frac{1}{(t-z_\alpha)^{k_\alpha}}\right\} = \omega_\alpha \qquad (\alpha \subset \mathfrak{M})$$

be a non-negative definite functional in E.

In the case where all $k_\alpha = 1$ it is possible to give a simple algebraic condition for the functional $\mathfrak{P}$ defined by the equalities

$$\mathfrak{P}\left(\frac{1}{(t-z_\alpha)}\right) = \omega_\alpha \qquad (\alpha \subset \mathfrak{M})$$

be non-negative definite. Indeed, if

$$g(t) = \sum_{j=1}^n \frac{A_j}{t - z_{\alpha_j}} + \sum_1^n \frac{\bar{A}_j}{t - \bar{z}_{\alpha_j}} \geq 0 \qquad (-\infty < t < \infty),$$

then

$$g(t) = \frac{\left|\sum_0^{n-1} \xi_k' t^k\right|^2}{\left|\prod_1^n (t - z_{\alpha_j})\right|^2} = \left|\sum_1^n \frac{\xi_j}{t - z_{\alpha_j}}\right|^2 = \sum_{j,k=1}^n \frac{\xi_j \bar{\xi}_k}{z_{\alpha_j} - \bar{z}_{\alpha_k}} \left\{\frac{1}{t - z_{\alpha_j}} - \frac{1}{t - \bar{z}_{\alpha_k}}\right\}.$$

Therefore $\mathfrak{P}$ is a non-negative definite functional if

$$\mathfrak{P}(g) = \sum_{j,k=1}^n \frac{w_{\alpha_j} - \bar{w}_{\alpha_k}}{z_{\alpha_j} - \bar{z}_{\alpha_k}} \xi_j \bar{\xi}_k \geq 0,$$

for any ξ_j and $\alpha_j \subset \mathfrak{M}$ $(j = 1, 2, \cdots, n)$. Thus, it follows from proposition A that:

B. *For there to exist an N-function $F(z)$ which satisfies the equations*

$$F(z_\alpha) = w_\alpha \quad (\Im z_\alpha > 0;\ \alpha \subset \mathfrak{M}),$$

it is necessary and sufficient that the Hermitian form

$$\sum_{j,k=1}^{n} \frac{w_{\alpha_j} - \bar{w}_{\alpha_k}}{z_{\alpha_j} - \bar{z}_{\alpha_k}} \xi_j \bar{\xi}_k \tag{48}$$

be non-negative for any $\alpha_j \subset \mathfrak{M}$ $(j = 1, 2, \cdots, n)$.

The last proposition was established in a somewhat narrower form (by an entirely different method) by G. Pick (Math. Ann. Bd. 77, 1916). Among other ways it may be obtained from Theorem 2. [1)]

We shall let the reader verify with the help of formula (47) that in case at least one of the non-negative forms (48) of some order n_0 is singular of rank $r < n_0$, then an N-function $F(z)$ is defined uniquely and is a real rational function with the degrees of its numerator and denominator not exceeding r and with only simple real poles with negative residues.

3. For further applications of Theorem 6 we introduce some conditions:

Let $F(z)$ be some N-function and x a real number. If the function $\sigma_1(t)$ of the representation (45) of the function $F(z)$ satisfies the condition that $\sigma_1(x+0) - \sigma_1(x-0) = 0$ and the improper integral

$$\int_{-\infty}^{\infty} \frac{1 + xt}{t - x}\, d\sigma_1(t)$$

exists, we shall consider that $F(x)$ is defined, as determined by the equality

$$F(x) = a_0 x + a_1 + \int_{-\infty}^{\infty} \frac{1 + xt}{t - x}\, d\sigma_1(t).$$

If, moreover the integral in the expression

$$a_0 + \int_{-\infty}^{\infty} \frac{1 + t_2}{(t - x)^2}\, d\sigma_1(t)$$

exists, the value of this expression shall be taken as the value of the derivative $F'(x)$.

We may now formulate the following proposition:

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1) Cf. M. Kreĭn and P. Rehtman [10].

C. *For there to exist an N-function $F(z)$ which satisfies the conditions*

$$\begin{aligned} F(x_\alpha) &= w_\alpha \\ F'(x_\alpha) &\le w'_\alpha \end{aligned} \qquad (\alpha \subset \mathfrak{M}), \tag{49}$$

where x_α, w_α, w'_α $(\alpha \subset \mathfrak{M})$ are given real numbers $(x_\alpha \neq x_\beta$ for $\alpha \neq \beta)$ it is necessary and sufficient that the symmetric forms

$$\sum_{j,k=1}^{n} h(\alpha_j, \alpha_k)\,\xi_j\,\xi_k \tag{50}$$

are non-negative for any $\alpha_j \subset \mathfrak{M}$ $(j = 1, 2, \cdots, n)$, where the numbers $h(\alpha, \beta)$ $(\alpha, \beta \subset \mathfrak{M})$ are defined by the equalities

$$h(\alpha, \beta) = \begin{cases} \dfrac{w_\alpha - w_\beta}{x_\alpha - x_\beta} & \text{for} \quad \alpha \neq \beta \\ w'_\alpha & \text{for} \quad \alpha = \beta. \end{cases}$$

In fact, setting

$$\mathfrak{P}\left\{\frac{1}{t - z_\alpha}\right\} = w_\alpha, \quad \mathfrak{P}\left\{\frac{1}{(t - z_\alpha)^2}\right\} = w'_\alpha,$$

we define a certain functional $\mathfrak{P}$ on the set $E = \left\{\dfrac{1}{t - z_\alpha}, \dfrac{1}{(t - z_\alpha)^2}\right\}$ $(\alpha \subset \mathfrak{M})$.

From the fact that every non-negative combination

$$g(t) = \sum_{1}^{n} \frac{A_j}{t - x_{\alpha_j}} + \sum_{1}^{n} \frac{B_j}{(t - x_{\alpha_j})^2} \ge 0 \qquad (-\infty < t < \infty)$$

admits a representation

$$g(t) = \Big(\sum_{1}^{n} \frac{\xi_j}{t - x_{\alpha_j}}\Big)^2 = \sum_{j,k=1}^{n} \frac{\xi_j\,\xi_k}{(t - x_{\alpha_j})(t - x_{\alpha_k})}$$

and

$$h(\alpha, \beta) = \mathfrak{P}\left\{\frac{1}{(t - x_\alpha)(t - x_\beta)}\right\} \quad (\alpha, \beta \subset \mathfrak{M}),$$

it follows that the non-negativeness of form (50) is a necessary and sufficient condition for the functional $\mathfrak{P}$ to be positive definite. It is easy to see that the functions

$$\omega_0(t) = \frac{1}{t^2 + 1}, \quad \omega_1(t) = \frac{1}{t^2 + 1}$$

are dominated by $\breve{E}$ as before and therefore, if $\mathfrak{P}$ is a positive definite functional, we may extend the definition of the quantities $\mathfrak{P}(\omega_i)$ $(i = 0, 1)$ so that the functional remains positive definite in the set $E_1 = E + \{\omega_0, \omega_1\}$. In the decomposition

$$\frac{1}{t - x_\alpha} = x_\alpha \omega_0(t) + \omega_1(t) + g_\alpha(t)$$

$g(t)$ is a normal function in E_1, since

$$\lim_{|t| \to \infty} \frac{g_\alpha(t)}{\omega_\alpha(t)} = 0 \quad \text{and} \quad \lim_{t \to x_\alpha} \frac{g_\alpha(t)}{\omega_\alpha(t)} = 0$$

for

$$\omega_\alpha(t) = \omega_0(t) + \frac{1}{(t - x_\alpha)^2}.$$

In the decomposition

$$\frac{1}{(t - x_\alpha)^2} = \omega_0(t) + f_\alpha(t)$$

the function $f_\alpha(t)$ is semi-normal, since

$$\lim_{|t| \to \infty} \frac{f_\alpha(t)}{\omega_0(t)} = 0, \quad \lim_{t \to x_\alpha} \frac{f_\alpha(t)}{\omega_0(t)} = +\infty.$$

Therefore by Theorem 6 there exist a non-decreasing function $\sigma(t)$ $(-\infty < t < \infty)$ and constants $a_0 \geq 0$, $a_1 \gneqq 0$ such that

$$\begin{cases} w_\alpha = \mathfrak{P}\left\{\dfrac{1}{t - x_\alpha}\right\} = a_0 x_\alpha + a_1 + \displaystyle\int_{-\infty}^{\infty} \left[\frac{1}{t - x_\alpha} - \frac{t}{t^2 - 1}\right] d\sigma(t) \\ \\ w'_\alpha = \mathfrak{P}\left\{\dfrac{1}{(t - x_\alpha)^2}\right\} \geq a_0 + \displaystyle\int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t - x_\alpha)^2} \qquad (\alpha \subset \mathfrak{M}). \end{cases} \tag{51}$$

Therefore, the N-function $F(z)$ defined by equality (45) satisfies conditions (49). Conversely, if an N-function $F(z)$ satisfies conditions (49) and therefore relation (51) holds, then by reasoning similar to the proof of proposition A we easily show that the functional $\mathfrak{P}$ is positive definite and hence the form (51) is non-negative.

Proposition C is proved.

Note that if one of the forms (50) is singular with rank r, then $F(z)$ will be a rational function of the same nature as in the corresponding case of proposition B.

It is curious to also note that if the points x_α are finite in number ($\mathfrak{M}$ is a finite set) and form (50) which contains all the x_α is a positive definite form, then the N-function $F(z)$ may be defined in an infinite number of ways such that the equal sign holds in (49).[1] It is possible to obtain this result by the same method by which in Theorem 7 we eliminated the constant $M > 0$, and by introducing the interval $\mathfrak{I}_\delta$ which is obtained from the infinite axis by deleting all of the neighborhoods $(x_\alpha - \delta, x_\alpha + \delta)$.

In concluding this article we note that the applications of our theorems in the theory of N-functions are not limited to propositions A, B, and C. If we wished, we could by using the methods developed in this article give entirely new solutions to a series of questions which were examined by Nevanlinna [34b].

1) This assertion, as well as proposition B, may be proved by purely algebraic methods; in this connection see Kotelyanskiĭ [8].

ON POSITIVE DEFINITE FUNCTIONALS IN LINEAR NORMED SPACES

M. KREĬN

In any linear normed space one can very simply establish a notion of inequality between elements. This in turn permits the introduction in such spaces of the notion of positive definite functionals. It turns out that theorems on the extension of positive definite functionals also hold here. These theorems can be considered as generalizations of certain of the results of §1 of the preceding paper. But they also present independent interest and can be used for different purposes. [1]

§1

1. A set S of certain elements $x, y, z, \cdots$ is called a *linear system* if on the elements of this system we can perform the basic linear operations –addition of elements and multiplication of elements by scalars– without going outside this set, and these operations are subject to the usual laws of algebra. About the scalars we shall always assume that they are arbitrary real numbers. A linear system R is called a *linear normed* (according to Banach) *space* if to each element (we shall say vector) $x \subset E$ is associated a real number $\| x \|$ such that the following requirements are satisfied:

1. $\| x \| > 0$ if $x \neq \theta$, [2]

2. $\| \lambda x \| = | \lambda | \, \| x \|$ (λ is any scalar), [3]

3. $\| x + y \| \leq \| x \| + \| y \|$ **(triangle axiom).**

In linear normed spaces the following notions are usually established:

By sphere $S(x_0, \rho)$ with center at the point x_0 and radius ρ is meant the set of all elements x satisfying the inequality

$$\| x - x_0 \| < \rho.$$

Each sphere $S(x_0, \rho)$ is called the *neighborhood* of the point x_0 of radius ρ.

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1) The present paper represents a translation with certain changes and additions of the author's paper [11 b]. From a different point of view L. V. Kantorovič [6 b] has conducted extensive research connected with the notion of inequalities in linear normed spaces.

2) θ is the zero of the linear system S.

3) From requirements 2 and 1 it follows that $\| \theta \| = \| 0 \cdot x \| = 0 \, \| x \| = 0$.

The notion of neighborhood allows the introduction of the notion of *limit* of a sequence of elements $\{x_n\}_1^\infty$, the notions of *interior*, *boundary*, and *exterior* of a point set $G \subset E$ and other analogous concepts. In addition bounded and unbounded sets are defined as usual.

If a certain operation $f(x)$ relates to every element $x \subset G$ $(G \subset E)$ a certain real number, then we say that the functional $f(x)$ is defined (or given) in G. If from

$$x = \lim x_n, \quad x \subset G, \quad \{x_n\}_1^\infty \subset G$$

follows

$$f(x) = \lim f(x_n),$$

then the functional $f(x)$ is called *continuous* in G.

In what follows we shall consider only those functionals $f(x)$ whose domain of definition G is a linear system, and consequently $G \subset E$ is a linear normed space (subspace of E).

Any such functional $f(x)$ is called *additive* if

$$f(x + y) = f(x) + f(y) \quad \text{for all} \quad x, y \subset G.$$

An additive functional $f(x)$ is called *linear* if it is continuous. It is easy to show that a linear functional is a *homogeneous* functional, i.e.,

$$f(\lambda x) = \lambda f(x) \qquad (x \subset G).$$

Let us show that:

In order that an additive functional $f(x)$ be linear it is necessary and sufficient that a real constant $M > 0$ exist such that

$$|f(x) \leq M \|x\| \qquad (x \subset G). \tag{1}$$

For the proof of the necessity of this condition let us assume the inverse, i.e., that the expression $f(x)/\|x\|$ is unbounded in G. Then for any natural number n there exists an element $x_n \subset G$ for which

$$|f(x_n)| > n \|x_n\|. \tag{2}$$

Let us put

$$y_n = \frac{x_n}{n \|x_n\|} \qquad (n = 1, 2, \cdots); \tag{3}$$

then $\|y_n\| < \frac{1}{n}$ and consequently $\lim y_n = \theta$.

Because of the continuity of the functional $f(x)$ we must have

$$\lim f(y_n) = f(\theta) = 0,$$

while by (2) and (3)

$$|f(y_n)| > 1 \qquad (n = 1, 2, 3, \cdots).$$

We have reached a contradiction.

Conversely, for some M let the additive functional satisfy inequality (1); then from

$$\lim x_n = x \qquad (x \subset G, \quad \{x_n\}_1^\infty \subset G)$$

follows $\lim f(x_n) = f(x)$, since

$$|f(x) - f(x_n)| = |f(x - x_n)| < M \|x - x_n\|.$$

In view of (1) the following quantity makes sense

$$\|f\|_G = \sup \frac{|f(x)|}{\|x\|} = \sup_{\|x\| \leq 1} |f(x)| \qquad (x \subset G).$$

This quantity is the smallest value M which fulfills the inequality (1), and is called the *norm* in G of the functional $f(x)$; if $G = E$, then we say simply norm of the functional f and write $\|f\|$.

2. A set $K \subset E$ will be called a *cone* if it has the following properties:

1°. If $x \subset K$, then $\lambda x \subset K$ for $\lambda \geq 0$.

2°. If $x, y \subset K$, then $x + y \subset K$.

3°. If $x \subset K$ and $x \neq \theta$, then $-x \subset E - K$.

4°. The set K contains some sphere.

Having settled on some fixed cone K, let us agree to write $x \geq y$ $(x > y)$ if $x - y \subset K$ ($x - y$ is an interior point of K). It is easy to convince oneself that the inequality defined in this way is subject to the usual laws of algebra. A functional $f(x)$ defined in a linear subspace $G \subset E$ containing elements of K is called *positive definite* if $f(x) \geq 0$ for $x \geq \theta$, i.e., $x \subset K$.

We note that every additive positive definite functional $f(x)$ defined on all of E [1] is a linear functional. Indeed, if $f(x)$ is a positive definite additive functional, then from the inequality $x \leq y$ follows the inequality $f(x) \leq f(y)$. On the other hand, by condition 4° the cone K contains a sphere $S(x_0, \rho)$, i.e.,

1) Or in a subspace $G \subset E$ containing interior points of the cone K, i.e., containing positive elements.

$$x \geq \theta \quad \text{for} \quad \| x - x_0 \| \leq \rho.$$

Therefore, for all vectors e with norm not greater than unity ($\| e \| \leq 1$) we have the inequality

$$x_0 \pm \rho e \geq \theta \quad \text{or} \quad \pm e \leq \frac{1}{\rho} x_0. \tag{4}$$

Hence

$$| f(e) | \leq f(\frac{x_0}{\rho}) = M,$$

and consequently

$$| f(x) | \leq M \| x \|,$$

since for every rational $r \geq \| x \|$, in view of the additivity of the functional $f(x)$

$$\frac{1}{r} | f(x) | = \left| f(\frac{x}{r}) \right| \leq M.$$

The assertion is proved.

Theorem 1. *If the subspace $G \subset E$ contains at least one positive element $x_0 > \theta$, then every linear positive definite functional $f(x)$ defined in G can be extended to a linear positive definite functional $F(x)$ defined in all of E.*

The phrase "can be extended ..." is to be understood in the sense that

$$F(x) = f(x) \quad \text{for} \quad x \subset G.$$

Proof. It is sufficient to consider only the case when $G \neq E$. Let $y_0 \subset E - G$. We denote by $x' \subset G$ and $x'' \subset G$ two arbitrary elements for which

$$x' \leq y_0 \leq x''. \tag{5}$$

Such elements x' and x'' exist since, putting the vector e in inequality (4) equal to $y_0 / \| y_0 \|$ we get that

$$-\frac{x_0}{\rho} \| y_0 \| \leq y_0 \leq \frac{x_0}{\rho} \| y_0 \|.$$

Since from (5) always follows $f(x') \leq f(x'')$, there exists a number ξ such that

$$\sup f(x') \leq \xi \leq \inf f(x''), \tag{6}$$

where the sup (or the inf) is extended over all x' (or all x'') satisfying inequality (5).

Let us consider the linear space $G_0 \supset G$ formed by all elements y of the form

$$y = x + t y_0,$$

where $x \subset G$ and t is an arbitrary real number.

Set

$$\phi(y) = f(x) + t\xi.$$

The functional $\phi(y)$ determined in this way in G_0 is additive and positive definite. The first is obvious; the second follows from the fact that if $y = x + ty_0 \geq \theta$ $(t \neq 0)$, then

$$-\frac{1}{t}x \begin{cases} \leq y_0 & \text{for } t > 0, \\ \geq y_0 & \text{for } t < 0, \end{cases}$$

and consequently by (6)

$$-\frac{1}{t}f(x) \begin{cases} \leq \xi & \text{for } t > 0, \\ \geq \xi & \text{for } t < 0. \end{cases}$$

Hence in both cases $\phi(y) = f(x) + t\xi \geq 0$.

In this way we see that it is sufficient to well-order the set $E - G$ in order by successive extensions of $f(x)$ with the help of the process described to arrive at the functional $F(x)$ whose existence we wished to show.[1)]

Theorem 2. *If a linear subspace $G \subset E$ has the property that the set of its unit elements e $(\| e \| = 1)$ is situated at positive distance d from K,[2)] then any linear functional $f(x)$ defined in G can be extended to a linear positive definite functional $F(x)$ defined in all of E.*[3)]

Proof. Let x_0 be any positive element $(x_0 > \theta)$. Let us show that the set H of all elements x such that

$$x \leq x_0, \quad x \subset G \tag{7}$$

is a bounded set. Indeed, from (7) it follows that

$$x_1 = x_0 - x \subset K,$$

and therefore

$$\frac{\| x_0 \|}{\| x \|} = \left\| \frac{x_1}{\| x \|} + \frac{x}{\| x \|} \right\| = \| x' - e \| \geq d,$$

---

1) Without particular difficulty it can be shown that if the space E is *separable*, i.e., there exists a countable set dense in it, then the functional $F(x)$ can be arrived at in a countable number of steps.

2) I.e., the distance $\| e - x \|$ of any vector $e \subset G$ $(\| e \| = 1)$ from any element $x \subset K$ has the positive lower bound d.

3) The condition indicated in the theorem for the subspace G is not only *sufficient*, but is also *necessary* in order that any linear functional defined in G admit a positive definite extension to E.

since

$$x' = \frac{x_1}{\|x\|} \subset K, \quad e = -\frac{x}{\|x\|} \subset G \quad \text{and} \quad \|e\| = 1.$$

We consider now the space $G_0 \supset G$, formed by all elements y of the form: $y = x + tx_0$, where $x \subset G$ and t is an arbitrary real number. Let us set in this space

$$\phi(y) = f(x) + t\xi,$$

where the number ξ is subject to the condition

$$\xi \geq \sup_{x \subset H} f(x),$$

or, what amounts to the same, to the condition

$$\xi \geq f(x) \quad \text{for} \quad x \lesssim x_0 \quad \text{and} \quad x \subset G. \tag{8}$$

The additive functional $\phi(y)$ is positive definite. Indeed, if $y = x + tx_0 \geq 0$ $(x \subset G,\ x \neq \theta)$, then $t > 0$, since in the contrary case we would have the inequality

$$x = x + tx_0 + |t|\, x_0 \geq \theta,$$

which is impossible. Therefore from the inequality

$$y = x + tx_0 \geq \theta \qquad (x \subset G)$$

for $y \neq \theta$ it follows that $t > 0$, $x_0 \geq -x/t$, and consequently in view of (8)

$$\xi \geq -\frac{1}{t} f(x) \quad \text{or} \quad \phi(y) = f(x) + t\xi \geq 0.$$

Since $x_0 > 0$ and $x_0 \subset G_0$, it follows from the first theorem that the functional $\phi(x)$, and with it the functional $f(x)$, can be extended to a linear positive definite functional $F(x)$ defined on all of E.

The theorem is proved.

3. A subspace $G \subset E$ is called finite-dimensional if in it there exist a finite number of elements $g_1, g_2, \cdots, g_n$ such that every element $x \subset G$ is represented in the form of a linear combination of the elements g_i $(i = 1, 2, \cdots, n)$, i.e.,

$$x = \sum_1^n \xi_i g_i.$$

If moreover the vectors g_i $(i = 1, 2, \cdots, n)$ are linearly independent, then the numbers ξ_i $(i = 1, 2, \cdots, n)$ are uniquely determined by the vector x and are called the *coordinates of the vector* x *in the basis* $g_1, g_2, \cdots, g_n$. The number

of basis elements $g_1, g_2, \cdots, g_n$ is easily seen not to depend on the basis and is called the *dimension number* of the space G.

To every vector $x = \Sigma_1^n \xi_i g_i$ of the n-dimensional space G we relate the vector $\mathfrak{x} = (\xi_1, \cdots, \xi_n)$ of euclidean space E_n in which the role of norm is played by the ordinary length, i.e.,

$$\| \mathfrak{x} \| = \sqrt{\sum_1^n \xi_i^2}.$$

This is obviously a one-to-one mapping of G onto E_n. We show that this mapping is continuous both ways, i.e., that from

$$\lim \mathfrak{x}_k = \mathfrak{x} \quad \text{follows} \quad \lim x_k = x \tag{9}$$

and conversely.

In fact, if $x = \Sigma_1^n \xi_i g_i$, then

$$\| x \| \leq \sum_1^n | \xi_i | \, \| g_i \| \leq \left(\sum_1^n \xi_i^2 \right)^{1/2} \left(\sum_1^n \| g_i \|^2 \right)^{1/2} = M \| \mathfrak{x} \|,$$

where

$$M = \left(\sum_1^n \| g_i \|^2 \right)^{1/2}$$

does not depend on x. Hence

$$\| x - x_k \| \leq M \| \mathfrak{x} - \mathfrak{x}_k \|$$

i.e., (9) is proved. In particular, from the proof it follows that $\| x \| = \chi (\xi_1, \cdots, \xi_n)$ is a continuous function of $\mathfrak{x} = (\xi_1, \cdots, \xi_n)$.

Let $\mu = \min \| x \|$ under the supplementary condition $\| \mathfrak{x} \| = (\Sigma_1^n \xi_i^2)^{1/2} = 1$. Since by the Weierstrass theorem the minimum is attained, $\mu > 0$. From the definition of μ it follows that

$$\mu \| \mathfrak{x} \| \leq \| x \| \quad \text{or} \quad \| \mathfrak{x} \| \leq \frac{1}{\mu} \| x \|. \tag{10}$$

Consequently,

$$\| \mathfrak{x} - \mathfrak{x}_k \| \leq \frac{1}{\mu} \| x - x_k \|,$$

i.e., relation (9) can be inverted.

From the continuity both ways of the mapping $x \longleftrightarrow \mathfrak{x}$ follows

Lemma. *The unit sphere* $\| x \| \leq 1$ $(x \subset G)$ *of a finite-dimensional space* G *is a set compact in itself.* [1]

---

1) A set $H \subset E$ is called *compact* if every infinite subset of it contains a sequence tending to some element $x \subset E$. If, moreover, this limit element x belongs to H, then we say that H is compact in itself.

Indeed, under the mapping $x \longleftrightarrow \xi$ the sphere S_G in question, being a closed set, transforms into some bounded closed set in euclidean space. But the latter, by the Bolzano-Weierstrass theorem, is compact in itself, and consequently the sphere S_G is also compact in itself.

We may now formulate the theorem:

Theorem 3. *If $G \subset E$ is a linear finite-dimensional space not having any point in common with K ($\overline{K}$ the closure of K) except the point $x = \theta$, then every linear functional $f(x)$ defined in G can be extended to a positive definite functional $F(x)$ defined on all of E.*

Proof. According to Theorem 2 it is enough to prove that the set of unit elements of G is removed from K by distance $d > 0$. Suppose that $d = 0$. Then there will exist a sequence $\{e_n\} \subset G$ ($\| e_n \| = 1$; $n = 1, 2, \cdots$) and a sequence $\{x_n\} \subset K$ such that $\| x_n - e_n \| \to 0$. In view of the preceding lemma the sequence $\{e_n\}$ contains some subsequence $\{e_{n_\nu}\}$ tending to a limit element $e \subset G$ ($\lim e_{n_\nu} = e$). But then $\lim x_{n_\nu} = e$ and consequently $e \subset \overline{K}$. But this is impossible since $e \neq \theta$ ($\| e \| = 1$), which proves the theorem.

4. Let us agree to say that K is a *sharp cone* if in it there exists a positive element u such that the set $H(u)$ of elements x satisfying the inequality

$$-u \leq x \leq u \tag{11}$$

is bounded.

Theorem 4. *If K is a sharp cone, then every linear functional $f(x)$ defined in E admits a decomposition*

$$f(x) = f_1(x) - f_2(x) \qquad (x \subset E),$$

where $f_1(x)$ and $f_2(x)$ are linear positive definite functionals satisfying the condition

$$\| f_1 \| + \| f_2 \| \leq C \| f \|, \tag{12}$$

with C a certain constant not depending on the functional f.

Proof. Let u be an element of K for which the set $H(u)$ is bounded. Put

$$\gamma = \sup_{x \subset H(u)} \| x \|. \tag{13}$$

Let us now form a new linear space $E^{(2)}$. As the elements of this space we will take all ordered pairs $Z = (x, y)$, where x and y are any elements of E and the product of an element $Z = (x, y)$ by a scalar λ is defined by the equality

$$\lambda Z = (\lambda x,\ \lambda y),$$

and the sum of two elements $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$ by the equality

$$Z_1 + Z_2 = (x_1 + x_2,\ y_1 + y_2).$$

As norm of Z let us take, for example, the quantity

$$\| Z \| = \| x \| + \| y \|.$$

Let us denote by $K^{(2)}$ the set of those $Z = (x, y)$ for which $x \geq \theta$ and $y \geq \theta$. It is easy to see that $K^{(2)}$ is a certain cone in the space $E^{(2)}$. Using the cone $K^{(2)}$ we introduce in the space $E^{(2)}$ the notion of inequality between elements Z and the notion of positive definite functional $F(Z)$.

Let us denote by G the linear space ($G \subset E^{(2)}$) consisting of all elements Z of the form

$$Z = X + \lambda U,$$

where $U = (u, u)$, λ is an arbitrary real scalar, and X is an arbitrary pair $(x, -x)$, where $x \subset E$.

Now let $f(x)$ be an arbitrary linear functional defined on all of E. With it we associate the functional $F(Z)$ defined in G by the equality

$$F(Z) = f(x) + \lambda\gamma \| f \|.$$

It is easily seen that the functional $F(Z)$ is additive in G; let us show that it is positive definite. Indeed, if $Z = X + \lambda U \geq \theta$, then by definition this means that

$$x + \lambda u \geq \theta \quad \text{and} \quad -x + \lambda u \geq \theta. \tag{14}$$

Adding these inequalities and recalling that $u > \theta$, we find that $\lambda \geq 0$. We are interested only in the case $\lambda > 0$ since for $\lambda = 0$ it follows from (14) that $x = \theta$.

For $\lambda > 0$ inequality (14) may be rewritten

$$-u \leq \frac{x}{\lambda} \leq u,$$

whence the element $\frac{x}{\lambda}$ belongs to the set $H(u)$ and consequently by definition (13) of the constant γ

$$\frac{1}{\lambda} |f(x)| = |f(\frac{x}{\lambda})| \leq \| f \| \cdot \| \frac{x}{\lambda} \| \leq \gamma \| f \|.$$

Hence

$$F(Z) = f(x) + \lambda\gamma \| f \| \geq 0.$$

Since the element $U = (u, u)$ is positive (U lies inside $K^{(2)}$), then Theorem 1

applies to the positive definite functional $F(Z)$.

Thus, there exists a positive definite functional $\Phi(Z)$ defined in all of $E^{(2)}$ and such that

$$\Phi(Z) = F(Z) \quad \text{for} \quad Z \subset G.$$

If $X = (x, -x)$, then

$$X = X' - X'',$$

where

$$X' = (x, \theta), \quad X'' = (\theta, x).$$

Consequently

$$f(x) = F(X) = F(X') - F(X'') = f_1(x) - f_2(x),$$

where

$$f_1(x) = F(X'), \quad f_2(x) = F(X'').$$

Since $F(Z)$ is a positive definite functional, then the additive functionals $f_1(x)$ and $f_2(x)$ are also positive definite functionals.

For the proof of the theorem it remains to show that for the indicated method of decomposition there exists a constant C (not depending on the functional f) for which inequality (12) is satisfied.

For this we first note that from the relation

$$(u, u) = (u, \theta) + (\theta, u)$$

follows the equality

$$\gamma \| f \| = F(U) = f_1(u) + f_2(u), \tag{15}$$

moreover, since $u > \theta$, it follows that $f_1(u) \geq 0$ and $f_2(u) \geq 0$.

In addition, the cone K must contain some sphere $S(u, \rho)$, i.e.,

$$u + \rho e \geq \theta \quad \text{if} \quad \| e \| = 1. \tag{16}$$

Hence, in view of the positivity of the functionals $f_i(x)$ $(i = 1, 2)$

$$f_i(u) \pm \rho f_i(e) \geq \theta \quad \text{or} \quad |f_i(e)| \leq \frac{1}{\rho} f_i(u) \qquad (i = 1, 2).$$

In other words

$$\| f_i \| \leq \frac{1}{\rho} f_i(u) \qquad (i = 1, 2).$$

Upon comparing this inequality with (15) we find that

$$\| f_1 \| + \| f_2 \| \leq \frac{\gamma}{\rho} \| f \|. \tag{17}$$

Remark. If the sphere $S(u, \rho)$ is contained in K, then from (16) it follows that the set $H(u)$ defined by the inequalities contains the sphere $S(\theta, \rho)$. In certain cases, by

choosing a suitable u and taking the largest possible ρ, it can be arranged that the set $H(u)$ coincides with the sphere $S(\theta, \rho)$. In this case the constant γ coincides with ρ and consequently inequality (17) becomes the following

$$\|f_1\| + \|f_2\| \leq \|f\|.$$

Since on the other hand we always have

$$\|f\| = \|f_1 + f_2\| \leq \|f_1\| + \|f_2\|,$$

in this case there is the equality

$$\|f_1\| + \|f_2\| = \|f\|.$$

In what follows we show that the cone K is sharp if and only if linear functionals $f(x)$ defined on all of E admit the decomposition indicated in Theorem 4.

§2

1. By T_a, where $a \subset E$, we shall denote the *translation* corresponding to the element a, i.e., the operation assigning to each element $x \subset E$ the element

$$T_a x = x + a.$$

It is obvious that

1. $T_a T_b x = T_b T_a x = x + a + b \quad (x \subset E)$.

2. $T_a x - T_a y = x - y$.

Following Mazur [33] we shall call the set $F \subset E$ a *linear manifold* if it can be obtained by some translation T_a from some linear system $G \subset E$ $(F = T_a G)$. Obviously, by translating any linear manifold we obtain again a linear manifold.

Let us denote by E^* the set of all linear functionals $f(x)$ defined on all of E.

A set $H \subset E$ is called a *hyperplane* if its equation can be represented in the form

$$f(x) = c, \tag{18}$$

where $f \subset E^*$ and c is a certain real number. Parallel with the hyperplane H we consider the hyperplane H_0 with equation

$$f(x) = 0.$$

Let a be some element of H, i.e., $f(a) = c$. If $x \subset H_0$, then $f(x + a) = f(x) + f(a) = f(a) = c$, i.e., $x' = x + a \subset H$; conversely, if $x' \subset H$, then $f(x' - a) = f(x') - f(a) = 0$, i.e., $x = x' - a \subset H_0$. Hence

$$H = T_a H_0.$$

Since H_0 is a linear system (linear space), it follows that H is a linear manifold. Thus, every hyperplane is a linear manifold.

Let us agree to say that the points x_1 and x_2 lie on *different sides* of the hyperplane H if the differences

$$f(x_1) - c \quad \text{and} \quad f(x_2) - c$$

have opposite signs; in the contrary case we say that the points x_1, x_2 lie on the same side of the hyperplane (18).

We call *segment joining* x_1 and x_2 the set of points z of the form

$$z = tx_1 + (1 - t)x_2,$$

where

$$0 \leq t \leq 1.$$

It is easy to see that if the points x_1 and x_2 lie on different sides of the hyperplane, then the segment joining them pierces the hyperplane, i.e., has in common with it one and only one point z for which $0 < t < 1$.

A set $\mathfrak{M}$ is called *convex* if together with any two points it also contains the segment joining them. A convex set $\mathfrak{K}$ is called a convex *body* if it is closed and contains interior points.

Theorem 5 (Ascoli–Mazur).[1] *If the linear manifold G does not contain interior points of the convex body $\mathfrak{K}$, then there exists at least one hyperplane H containing G on one side of which is situated the convex body $\mathfrak{K}$.*

Proof. If the same translation is applied to both the manifold G and the body $\mathfrak{K}$ and the theorem is proved for the manifold and body displaced in this way, then this same theorem will be proved for the given G and $\mathfrak{K}$. Therefore without loss of generality we may suppose that G contains the origin θ, i.e., that G is a linear space.

Let us denote by $\mathfrak{K}_0$ the set of all interior points of $\mathfrak{K}$ and consider the smallest set

$$K = \mathop{\mathfrak{S}}_{0 \leq \lambda < \infty} \lambda \mathfrak{K}_0,$$

containing all the homothetic bodies $\lambda \mathfrak{K}_0$ $(0 \leq \lambda < \infty)$.

---

1) For finite dimensional spaces this theorem was already proved by Minkowski; for the case of linear separable spaces it was proved by Ascoli [19]. In the form in which we have formulated the theorem it was proved in another way by S. Mazur [33].

It is easy to see that K has the characteristic properties of a cone 1° and 4° (see p. 156). It also has property 2°. In fact, given $x_1 \subset K$ and $x_2 \subset K$, one can find $y_1 \subset \mathfrak{R}_0$ and $y_2 \subset \mathfrak{R}_0$ such that

$$x_1 = \lambda_1 y_1, \quad x_2 = \lambda_2 y_2 \qquad (\lambda_1 \geq 0, \ \lambda_2 \geq 0).$$

We may suppose that $\lambda_1 + \lambda_2 > 0$. Since $\mathfrak{R}_0$ is convex, then the point

$$y = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2} \subset \mathfrak{R}_0,$$

and consequently the point $x_1 + x_2 = (\lambda_1 + \lambda_2)\, y \subset K$.

In order to verify that the set K is a cone it remains to show that it has property 3°. Let us assume the contrary, that some $a \neq \theta$ has the property that $a \subset K$ and $-a \subset K$. Then one can find numbers $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$b_1 = \frac{a}{\lambda_1} \subset \mathfrak{R}_0 \quad \text{and} \quad b_2 = -\frac{a}{\lambda_2} \subset \mathfrak{R}_0.$$

Since $\mathfrak{R}_0$ consists only of interior points, it follows that $\mathfrak{R}_0$ contains also certain spheres $S_1 = S(b_1, \rho_1)$ and $S_2 = S(b_2, \rho_2)$. Consider the set S consisting of points z of the form

$$z = \frac{\lambda_1 x + \lambda_2 y}{\lambda_1 + \lambda_2} \qquad \text{where} \qquad x \subset S_1 \quad \text{and} \quad y \subset S_2.$$

It is easy to see that S is a sphere with center at the point θ and radius

$$\rho = \frac{\lambda_1 \rho_1 + \lambda_2 \rho_2}{\lambda_1 + \lambda_2}.$$

On the other hand, since $S_1 \subset \mathfrak{R}_0$ and $S_2 \subset \mathfrak{R}_0$, it follows that S also is contained in $\mathfrak{R}_0$, whence the point θ is an interior point of $\mathfrak{R}$, which contradicts the assumption.

Thus, K is a cone. With its aid we introduce the notion of inequality between elements and the notion of positive definite functional.

Let us take in K some element $x_0 > \theta$ and consider the space G_0 composed of all elements y of the form

$$y = x + tx_0,$$

where $x \subset G$, and t is an arbitrary real number.

Let us put in this space

$$f(y) = t.$$

The functional $f(y)$ defined in this way is positive definite. In fact, from the inequality $y = x + tx_0 \geq \theta$ it follows that $t \geq 0$, since in the contrary case ($t < 0$) we would have $x \geq |t| x_0 > \theta$, which is impossible since $x \subset G$ is not an interior point of the body $\mathfrak{K}$, and accordingly not of the cone K. Consequently, if $y = x + tx_0 \geq \theta$, then $f(y) = t \geq 0$.

Since the linear space G_0 contains a positive element x_0, by Theorem 1 there exists a positive definite functional $F(x)$ such that

$$F(x) = f(x) \quad \text{for} \quad x \subset G.$$

Consequently, the hyperplane

$$F(x) = 0$$

contains the manifold G; since, moreover, the body $\mathfrak{K}$ lies on one side of this hyperplane (because, if $x \subset \mathfrak{K}$, then $x \geq \theta$ and accordingly $F(x) > 0$). The theorem is proved.

Theorem 6 (Hahn). *Let $G \subset E$ be some linear subspace. Every linear functional $f(x)$ defined in G can be extended to a linear functional $F(x)$, defined on all of E, and with the same norm, i.e., such that*

$$\| F \| = \| F \|_E = \| f \|_G.$$

Proof. Assuming that $f(x) \not\equiv 0$ ($x \subset G$), i.e., that $\| f \|_G > 0$, let us consider in G the hyperplane L with equation

$$f(x) = \| f \|_G \qquad (x \subset G). \tag{19}$$

The hyperplane L, considered in E, represents a certain linear manifold. This manifold does not contain any interior point of the unit sphere $S(\theta, 1)$. Indeed, by definition

$$f(x) \leq \| f \|_G \quad \text{if} \quad \| x \| \leq 1$$

and therefore if $\| a \| < 1$, then clearly $f(a) < \| f \|_G$.

By the theorem of Ascoli–Mazur the manifold L is contained in a hyperplane H, on one side of which is situated the unit sphere; let its equation be

$$F(x) = c. \tag{20}$$

Since the point θ does not belong to H, we have $c \neq 0$. Therefore we always can normalize equation (20) in such a way that $c = \| f \|_G$. Then since $H \supset L$

$$F(x) = f(x) = \| f \|_G \quad \text{for} \quad x \subset L.$$

Hence it follows that

$$F(x) = f(x) \quad \text{for any} \quad x \subset G. \tag{21}$$

Indeed, take any element a of L. For an arbitrary $x \subset G$ we can always find λ such that $x - \lambda a \subset L$; but then

$$F(x - \lambda a) = f(x - \lambda a).$$

Since $f(a) = F(a)$, (21) follows.

In view of (21)

$$\| F \| = \sup_{x \subset E} | F(x) | \geq \sup_{x \subset G} | f(x) | = \| f \|_G. \tag{22}$$

On the other hand, since the sphere $S(\theta, 1)$ lies on the same side of H as θ, we find considering, that in (20) $c = \| f \|_G$, that

$$F(x) \leq \| f \|_G \quad \text{for} \quad \| x \| \leq 1.$$

Comparing this with (22) we reach the conclusion that $\| F \| = \| f \|_G$. This proves the theorem.

Let us pause to outline the geometric interpretation of the theorem we have proved.

The hyperplane H is called a *supporting hyperplane* for a certain set $\mathfrak{M}$ if the latter is situated on one side of H and the distance from H to this set equals zero. [1]

If $g(x)$ is a linear functional, defined in E, then the hyperplane

$$g(x) = \| g \|,$$

as is easily seen, is a supporting hyperplane for the unit sphere $S(\theta, 1)$.

Thus L, considered in G, is a supporting hyperplane for the unit sphere of the subspace G. Hahn's theorem is obtained as a direct consequence of the fact that L is contained in a hyperplane of support for the unit sphere of the space E.

We note the following important consequence of Hahn's theorem.

Theorem 7. *For any element $a \subset E$ there exists some linear functional $f \subset E^*$ such that*

$$f(a) = \| a \|, \quad \| f \| = 1.$$

Proof. Consider the one-dimensional space G consisting of all vectors x of the form: $x = ta$ (t is any real number).

---

1) I.e., $\inf | x - y |$, where $x \subset \mathfrak{M}$, $y \subset H$ equal zero.

In this space we set

$$\phi(x) = t\|a\|,$$

so that, in particular, $\phi(a) = \|a\|$. Obviously $\|\phi\|_G = 1$. Upon extending the functional $\phi(x)$ (with the same norm) to a functional $f(x)$ defined on all of E, we obtain the desired functional.

We note that any functional $f(x)$ satisfying the conditions of Theorem 7 defines a supporting hyperplane $f(x) = \|a\|$ for the sphere $\|x\| \leq \|a\|$.

Among other things Theorem 7 shows that the set E^* of all linear functionals defined in E contains other functionals besides the functional θ^* everywhere equal to zero. It is not difficult to show how to define in a natural way the operations of addition and multiplication of functionals by scalars. We have already defined the norm of a functional. In view of all this, the set E^* also may be considered as a certain linear normed space. This space is called the *conjugate* space to the space E.

2. We return again to consider the cone K, introduced in §1. Theorem 7 allows us to prove the following refinement of Theorem 4.

Theorem 8. *In order that a cone K be a sharp cone it is necessary and sufficient that every functional $f \subset E^*$ admit a decomposition*

$$f(x) = f_1(x) - f_2(x) \quad (x \subset E),$$

where $f_1(x)$ and $f_2(x)$ are linear positive definite functionals satisfying the condition

$$\|f_1\| + \|f_2\| \leq C\|f\|, \tag{23}$$

in which C is a certain constant not depending on f.

Proof. Sufficiency of the condition was established in Theorem 4.

For the proof of necessity of the condition we choose in K an arbitrary positive element u.

Now let $x \subset \Pi(u)$, i.e.,

$$u \pm x \geq \theta. \tag{24}$$

We construct for the element x a functional $f \subset E^*$ such that

$$f(x) = \|x\|, \quad \|f\| = 1.$$

Let $f = f_1 - f_2$ be a decomposition of the functional f, of the sort discussed in Theorem 7. Then in view of (23),

$$\|f_1\| \leq C, \quad \|f_2\| \leq C. \tag{25}$$

Moreover, obviously

$$\| x \| = f(x) \leq | f_1(x) | + | f_2(x) |.$$

Consequently, for some i $(i = 1, 2)$

$$| f_i(x) | \geq \frac{1}{2} \| x \|. \tag{26}$$

Since the functional $f_i(x)$ is positive definite, it follows from (24) that

$$f_i(u) \pm f_i(x) \geq 0, \quad \text{i.e.,} \quad | f_i(x) | \leq f_i(u). \tag{27}$$

Upon comparing (25), (26), and (27) we find

$$\| x \| \leq 2 | f_i(x) | \leq 2 f_i(u) \leq 2 C \| u \|.$$

Thus, the set $H(u)$ is bounded, which is what was required to prove.

Since we chose the positive element u arbitrarily, we have in fact proved the proposition.

Theorem 9. *If the cone K is sharp, then no matter how the positive*[1] *element u is chosen, the set of elements x satisfying the inequality*

$$-u \leq x \leq u$$

is bounded.

§3

Let us see to what the results obtained lead in certain concrete spaces.

1. Let Q be an arbitrary abstract set, the variable elements of which we denote by q.

Let R be a linear normed space, the elements of which are certain bounded functions $x(q)$ (assigning to each q a certain real number) with norm defined by the equality

$$\| x \| = \sup_{q \subset Q} | x(q) |. \tag{28}$$

Assuming that the space R contains non-negative functions $x(q) \geq 0$ $(q \subset Q)$, we call a functional $f \subset R^*$ *positive definite* if on each such function it takes a non-negative value: $f(x) \geq 0$.

---

1) The requirement of positivity of the element u can be omitted, but this is of no interest.

Theorem 10. [1] *For the space R defined above every linear functional $f \subset R^*$ admits a representation*

$$f(x) = f_1(x) - f_2(x) \quad (x \subset R),$$

where $f_1(x)$ and $f_2(x)$ are positive definite functionals and

$$\| f \| = \| f_1 \| + \| f_2 \|.$$

Proof. Let us denote by M_Q the space of all possible bounded functions $x(q)$ $(q \subset Q)$ with the same definition of norm (28).

We extend a functional $f(x)$ given in R to the whole space M_Q with the same norm. In order not to complicate the notation we denote the extended functional also by the letter f. In addition, we denote by K the set of all non-negative functions $x(q) \subset M_Q$. It is easy to see that K is a certain cone in the space R.

Moreover, if we consider the function $u(q) \equiv 1$ $(q \subset Q)$, then the set $H(u)$ (see (11) §1) consists of functions satisfying the inequality

$$-1 \leq x(q) \leq 1,$$

and, consequently, coincides with the unit sphere $\| x \| \leq 1$. Therefore from Theorem 4 and the note after it follows the existence of a decomposition

$$f(x) = f_1(x) - f_2(x) \quad (x \subset M_Q), \tag{29}$$

where f_1 and f_2 are functionals positive definite in M_Q, and in M_Q

$$\| f \| = \| f_1 \| + \| f_2 \|.$$

Since $R \subset M_Q$, we have

$$\| f \|_R = \| f \| \geq \| f_1 \|_R + \| f_2 \|_R; \tag{30}$$

on the other hand, from (27) it follows that

$$\| f \|_R \leq \| f_1 \|_R + \| f \|_R. \tag{31}$$

Upon comparing (30) and (31) we realize that in these inequalities only the equality sign is possible. This proves the theorem.

2. In the general case of a space R the positive definite functionals, and

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1) In a certain more restrictive form this theorem may be found in Banach's book [20] (pp. 217–219). Banach proves it by special reasoning, whereas for us Theorem 10 is a special case of Theorem 4.

with them all linear functionals $f \subset R^*$, can be represented analytically in the form of a certain Fréchet integral. We limit consideration to the elementary case where Q is a closed interval $<a, b>$ and $R = C(a, b)$ is the set of functions $\phi(t)$ continuous in $<a, b>$ (now in place of $x(q)$ it is convenient to write $\phi(t)$, where $a \leq t \leq b$).

As we know from Theorem 1 §1 of Article II, every positive definite functional $f(\phi)$ defined in $C(a, b)$ admits a representation

$$f(\phi) = \int_a^b \phi(t)\, d\sigma(t), \tag{32}$$

where $\sigma(t)$ is some non-decreasing function. Obviously

$$\| f \| = \sigma(b) - \sigma(a).$$

Upon applying Theorem 9 we obtain Riesz's theorem which asserts that any linear functional $f(\phi)$ defined in $C(a, b)$ admits a representation

$$f(\phi) = \int_a^b \phi(t)\, d\sigma_1(t) - \int_a^b \phi(t)\, d\sigma_2(t),$$

where $\sigma_1(t)$ and $\sigma_2(t)$ are non-decreasing functions and moreover

$$\| f \| = \sigma_1(b) - \sigma_1(a) + \sigma_2(b) - \sigma_2(a).$$

It can also be shown directly with the help of Hahn's theorem that formula (32), where $\sigma(t)$ is now some function of bounded variation, exhausts all linear functionals in $C(a, b)$ and moreover one always has

$$\| f \| = \operatorname*{var}_{a \leq t \leq b} \sigma(t).$$

It is curious to note that in case $E = C(a, b)$ Theorems 1, 3 of the present article reduce to Theorems 1, 2 of Article II.

We remark that these same geometric considerations relating to smallest convex bodies which we used in the second proof of Theorem 1 (Article II) can be generalized to the case of any linear normed complete space. We refer for proofs to the corresponding article by the author [11a], and give here an outline of the essence of this generalization.

A linear normed space E is called *complete* if every sequence $\{x_n\} \subset E$ satisfying the condition

$$\| x_m - x_n \| \to 0 \text{ for } m, n \to \infty$$

has a limit.

Let E be a complete linear normed space. If to every value t of the interval $<a, b>$ corresponds a certain element $x_t \subset E$, then we shall say that in E there is given the curve L with equation: $x = x_t\ (a \leq t \leq b)$. This curve is called *weakly continuous* if for any $f \subset E^*$ the function $f(x_t)$ is a continuous function of t. If the curve L is weakly continuous, then to every function $\sigma(t)$ of bounded variation there corresponds one and only one element $y \subset E$ such that

$$f(y) = \int_a^b f(x_t)\, d\sigma(t). \tag{33}$$

For this element we write [1]

$$y = \int_a^b x_t\, d\sigma(t). \tag{34}$$

It turns out that the set of all $y \subset E$ admitting a representation (34) in which $\sigma(t)$ is a non-decreasing function and $\sigma(b) - \sigma(a) = 1$, coincides with the smallest closed convex body containing the curve L.

The following analogue of Theorem 1, Article II is proved without difficulty.

Theorem. *Let the weakly continuous curve L have the property that for at least one linear functional $f_0 \subset E^*$ the function*

$$f_0(x_t) > 0 \quad (a \leq t \leq b).$$

Then in order that the element y admit the representation (34) *in which $\sigma(t)$ is a non-decreasing function it is necessary and sufficient that the inequality*

$$f(x_t) \geq 0 \quad (a \leq t \leq b)$$

should imply the inequality

$$f(y) \geq 0$$

for every functional $f \subset E^*$.

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1) If the curve is continuous in the usual sense, i.e., $\lim_{t \to t_0} x_t = x_{t_0}\ (a \leq t_0 \leq b)$, then it is not difficult to show that the integral which we have defined coincides with the limit, which exists in this case,

$$\lim \sum_1^k x_{\xi_i} [\sigma(t_{i+1}) - \sigma(t_i)],$$

where $a = t_1 \leq \xi_1 \leq t_2 \leq \xi_2 \leq t_2 \leq \cdots \leq t_n \leq \xi_n \leq t_{n+1} = b$ and $\max | t_{i+1} - t_i | \to 0$.

Let us prove this theorem. Necessity follows from (33). We prove sufficiency.

We denote by G the linear set of continuous functions $\phi(t)$ $(a \leq t \leq b)$ admitting the representation $\phi(t) = f(x_t)$ $(f \subset E^*)$. We define on this set the functional $\Phi(\phi)$ by putting

$$\Phi(\phi) = f(y), \text{ if } \phi(t) = f(x_t).$$

By the conditions of the theorem the functional $\Phi(\phi)$ is positive definite (in the terminology of Article II non-negative definite), and since G contains the positive function $f_0(x_t)$, Theorem 1 of Article II applies to it. Thus, to it corresponds a non-decreasing function $\sigma(t)$ such that

$$\Phi(\phi) = f(y) = \int_a^b f(x_t)\, d\sigma(t)$$

for any function $\phi \subset G$, and consequently, for any functional $f \subset E^*$.

THE L-PROBLEM IN AN ABSTRACT LINEAR NORMED SPACE

M. KREĬN

§1

Henceforth we shall denote by the letter E a linear normed space. [1]

We shall be interested in the following two problems:

A. *Given n linearly independent elements $x_1, x_2, \cdots, x_n$, on E. Find necessary and sufficient conditions on the numbers $c_1, c_2, \cdots, c_n$, $L\,(\Sigma_1^n c_i^2 > 0,\ L > 0)$, in order for there to exist a linear functional* [2] *$f(x)$ which satisfies the relations*

$$f(x_i) = c_i,\ \|f\| \leq L,\ (i = 1, 2, \cdots, n). \tag{1}$$

B. *Given n linearly independent elements $x_1, x_2, \cdots, x_n$ from E. Find*

$$\frac{1}{\lambda} = \inf \|\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n\| \tag{2}$$

with the additional condition

$$c_1 \xi_1 + c_2 \xi_2 + \cdots + c_n \xi_n = 1,$$

where the numbers $c_1, c_2, \cdots, c_n$ $(\Sigma_1^n c_i^2 > 0)$ are given.

First of all we note that in problem B the lower bound is attained. In fact, let

$$\frac{1}{\lambda} = \lim_{k \to \infty} \|\xi_1^{(k)} x_1 + \xi_2^{(k)} x_2 + \cdots + \xi_n^{(k)} x_n\|,$$

where

$$c_1 \xi_1^{(k)} + c_2 \xi_2^{(k)} + \cdots + c_n \xi_n^{(k)} = 1 \quad (k = 1, 2, 3, \cdots).$$

Since the elements

$$x^{(k)} = \xi_1^{(k)} x_1 + \xi_2^{(k)} x_2 + \cdots + \xi_n^{(k)} x_n$$

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1) As regards the definitions and basic properties of linear normed spaces cf. §§1, 2 of the preceding article. We shall make use of only a small portion of the contents of those sections in these paragraphs. We note also that this article is with minor changes a set of lectures read in the winter of 1936 at the Scientific Research Institute of Mathematics and Mechanics in Kharkov.

2) When we speak simply of a "linear functional $f(x)$" we shall always have in mind a linear functional defined in all of E, i.e., an element of the conjugate space E^*.

obviously form a bounded and hence [1] compact set in the n-dimensional linear envelope E_n of the vectors $x_1, x_2, \cdots, x_n$ (E_n is the collection of all possible vectors x of the form $\xi_1 x_1 + \cdots + \xi_n x_n$), it follows that for some k_m ($m = 1, 2, \cdots$), there will exist a limit

$$\lim_{m \to \infty} (\xi_1^{(k_m)} x_1 + \cdots + \xi_n^{(k_m)} x_n) = \xi_1^0 x_1 + \cdots + \xi_n^0 x_n.$$

But from this equality we have [2]

$$\| \xi_1^0 x_1 + \cdots + \xi_n^0 x_n \| = \lim_{m \to \infty} \| \xi_1^{(k_m)} x_1 + \cdots + \xi_n^{(k_m)} x_n \| = \frac{1}{\lambda},$$

and also

$$\xi_j^0 = \lim_{m \to \infty} \xi_j^{(k_m)} \quad (j = 1, 2, \cdots, n),$$

and therefore,

$$c_1 \xi_1^0 + c_2 \xi_2^0 + \cdots + c_n \xi_n^0 = 1.$$

Thus, our assertion is proved.

The number λ defined by (2) will be denoted by $\lambda(c_1, c_2, \cdots, c_n)$.

According to (2), the function $\lambda = \lambda(c_1, c_2, \cdots, c_n)$ may be defined also with the help of the equality

$$\lambda(c_1, c_2, \cdots, c_n) = \max_{\xi_i} \frac{\sum_{i=1}^{n} c_i \xi_i}{\| \sum_{i=1}^{n} \xi_i x_i \|} = \max_{\| \sum_{i=1}^{n} \xi_i x_i \| = 1} \sum_{i=1}^{n} c_i \xi_i.$$

From this it is not difficult to conclude that

1°. $\lambda(c_1, c_2, \cdots, c_n) > 0$, if $\sum_{i=1}^{n} c_i^2 > 0$,

2°. $\lambda(tc_1, tc_2, \cdots, tc_n) = |t| \lambda(c_1, c_2, \cdots, c_n)$,

3°. $\lambda(c_1 + c_1', c_2 + c_2', \cdots, c_n + c_n') \leq \lambda(c_1, c_2, \cdots, c_n) + \lambda(c_1', c_2', \cdots, c_n')$.

We shall introduce into consideration the space E_n of vectors $c = (c_1, c_2, \cdots, c_n)$.

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1) Cf. the lemma §2 of the previous article.

2) Cf. the proof of the lemma §2 of the previous article.

By virtue of 1°, 2°, 3° we can define in this space the norm $\| c \|$ by the equality [1]

$$\| c \| = \lambda(c_1, c_2, \cdots, c_n). \tag{3}$$

But therefore, [2] by the finite dimensionality of the space E_n, two constants $\mu > 0$ and $M > 0$ can be found such that

$$\mu\sqrt{c_1^2 + c_2^2 + c_n^2} < \lambda(c_1, c_2, \cdots, c_n) < M\sqrt{c_1^2 + c_2^2 + \cdots + c_n^2} \tag{4}$$

and $\lambda(c_1, c_2, \cdots, c_n)$ will be a continuous function of the coordinates.

Consider the set of points $\mathfrak{R}_n$ in E_n defined by the inequality $\lambda(c_0, c_1, \cdots, c_n) \leq 1$.

We shall show that:

1°. $\mathfrak{R}_n$ is a convex, bounded body, i.e., a convex, bounded, closed set having interior points.

In fact, it follows from (4) that $\mathfrak{R}_n$ is contained in the Euclidean sphere

$$\sum_1^n c_i^2 \leq \frac{1}{\mu^2},$$

and on the other hand contains the sphere

$$\sum_1^n c_i^2 \leq \frac{1}{M^2}.$$

By the continuity of the function $\lambda(c_1, c_2, \cdots, c_n)$ the set $\mathfrak{R}_n$ is closed. Finally, if $\lambda(c_1, c_2, \cdots, c_n) \leq 1$, $\lambda(c_1', c_2', \cdots, c_n') \leq 1$, then in agreement with 2°, 3°,

$$\lambda(tc_1 + \overline{1-t}c_1', \cdots, tc_n + \overline{1-t}c_n') \leq t\lambda(c_1, \cdots, c_n) + \\ + (1-t)\lambda(c_1', \cdots, c_n') \leq 1 \quad (0 \leq t \leq 1),$$

in other words, if the points P and P' belong to $\mathfrak{R}_n$ then the segment $tP + (1-t)P'$ $(0 \leq t \leq 1)$ joining them belongs to $\mathfrak{R}_n$, i.e., $\mathfrak{R}_n$ is convex.

2°. The system (1) *has a solution if and only if* $L \geq \lambda(c_1, \cdots, c_n)$ (*basic*

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1) If the reader recalls the definition of a conjugate space it will become clear (cf. proof of Theorem 1) that the space E_n with its norm defined by (3) is nothing but the conjugate space to E_n (E_n is the linear envelope of vectors $x_1, x_2, \cdots, x_n$).

2) Cf. the proof of the lemma of §2 of the previous article.

proposition).

In fact, if the linear functional $f(x)$ is a solution of system (1) then for any ξ_i $(i = 1, 2, \cdots, n)$

$$\left|\sum_1^n c_i\xi_i\right| = \left|f\left(\sum_1^n \xi_i x_i\right)\right| \leqslant \|f\| \cdot \left\|\sum_1^n \xi_i x_i\right\| \leqslant L\left\|\sum_1^n \xi_i x_i\right\|$$

and therefore

$$\frac{1}{L} \leq \min_{\xi_i} \frac{\left\|\sum_1^n \xi_i x_i\right\|}{\left|\sum_1^n c_i \xi_i\right|} = \min_{\sum_1^n c_i \xi_i = 1} \left\|\sum_1^n \xi_i x_i\right\| = \frac{1}{\lambda}.$$

Now let $L \geq \lambda(c_1, c_2, \cdots, c_n)$. Consider the n-dimensional space E_n which consists of elements x of the form

$$x = \sum_1^n \xi_i x_i.$$

Let

$$\phi(x) = \sum_1^n c_i \xi_i;$$

$\phi(x)$ is a linear functional in E_n with norm

$$\|\phi\|_{E_n} = \sup_{x \subset E_n} \frac{|\phi(x)|}{\|x\|} = \sup_{\xi_i} \frac{\left|\sum_1^n c_i \xi_i\right|}{\left\|\sum_1^n \xi_i x_i\right\|} = \lambda(c_1, c_2, \cdots, c_n).$$

According to Hahn's theorem there exists a linear functional $f_0(x)$ in E such that

$$\|f_0\| = \|\phi\|_{E_n} = \lambda \leq L$$

and $f_0(x) = \phi(x)$ for $x \subset E_n$, in particular,

$$f_0(x_i) = \phi(x_i) = c_i \quad (i = 1, 2, \cdots, n).$$

Thus, the linear functional $f_0(x)$ is a solution of problem A. This proves our assertion.

Now let $\{x_i\} \subset E$ be some fixed sequence of linearly independent elements;

set

$$\lambda(c_1, c_2, \cdots, c_m) = \max_{\left\|\sum_1^m \xi_i x_i\right\| = 1} \sum_1^m c_i \xi_i \quad (m = 1, 2, 3, \cdots).$$

To properties 1°, 2°, 3° of the function $\lambda(c_1, c_2, \cdots c_n)$ we may now add the following:

4°. $\lambda(c_1, c_2, \cdots, c_n) \leq \lambda(c_1, c_2, \cdots, c_m)$ *for* $m > n$.

Defining E_m and $\mathfrak{R}_m$ $(m = 1, 2, \cdots)$ analogously to the way in which E_n and $\mathfrak{R}_n$ were defined earlier, for $m > n$ we shall agree to consider E_n as a subspace of E_m identifying every point $(c_1, \cdots, c_n) \subset E_n$ with the point $(c_1, \cdots, c_n, 0, \cdots, 0) \subset E_m$.

3°. *The projection of the body* $\mathfrak{R}_m$ *on* E_n $(m > n)$ *coincides with* $\mathfrak{R}_n$.

In fact, if the point $(c_1, \cdots, c_m) \subset \mathfrak{R}_m$, then $\lambda(c_1, \cdots, c_m) \leq 1$ and thus according to 4°, $\lambda(c_1, c_2, \cdots, c_n) \leq 1$, i.e., the point $(c_1, \cdots, c_n, \overbrace{0, \cdots, 0}^{m-n}) \subset \mathfrak{R}_m$. Thus the projection of $\mathfrak{R}_m$ on E_n is contained in $\mathfrak{R}_n$.

On the other hand, in accordance with proposition 1 there always exists a linear functional $f(x)$ such that

$$f(x_k) = c_k \quad (k = 1, 2, \cdots, n), \quad \|f\| = \lambda(c_1, \cdots, c_n).$$

With the help of this functional we define the numbers c_k' $(k = n + 1, \cdots, m)$ by the equalities

$$c_k' = f(x_k) \qquad (k = n + 1, \cdots, m).$$

According to the same proposition 1, $\lambda(c_1, \cdots, c_n) = \|f\| \geq \lambda(c_1, \cdots, c_n, c_{n+1}', \cdots, c_m')$, which together with property 4°, leads us to the equality

$$\lambda(c_1, c_2, \cdots, c_n) = \lambda(c_1, c_2, \cdots, c_n, c_{n+1}', \cdots, c_m').$$

Hence, if $\lambda(c_1, \cdots, c_n) \leq 1$, then $\lambda(c_1, \cdots, c_n, c_{n+1}', \cdots, c_m') \leq 1$, i.e., every point of $\mathfrak{R}_n$ is the projection of some point of $\mathfrak{R}_m$.

Thus the proposition is proved.

4°. *If the elements* $x_1, x_2, \cdots, x_n, y$ *are linearly independent and* $L > \lambda(c_1, \cdots, c_n)$, *then the values* η *for which the system*

$$\left.\begin{aligned} &f(x_i) = c_i \quad (i = 1, 2, \cdots, n), \quad f(y) = \eta \\ &\|f\| \leq L \end{aligned}\right\} \tag{5}$$

has a solution $f \subset E^*$, *fill some closed interval* (η', η''); *and moreover, if* $\eta = \eta'$ *or* η'', *then for every solution* f *of the system* (5) *the equality* $\|f\| = L$ *holds.*

Without loss of generality we may suppose $y = x_{n+1}$ and $\lambda(c_1, \cdots, c_n) = 1$, and therefore $L > 1$. We shall consider the convex body $\mathfrak{R}'_{n+1}$, defined by the inequality

$$\lambda(c_1, \cdots, c_n, \eta) \leq L.$$

It is obvious that $\mathfrak{R}'_{n+1}$ is obtained from $\mathfrak{R}_{n+1}$ by its expansion by a factor $L(>1)$. According to the previous proposition we may determine a point $P(c_1, c_2, \cdots, c_n, c^0_{n+1})$ so that $\lambda(c_1, c_2, \cdots, c_n, c^0_{n+1}) = \lambda(c_1, c_2, \cdots, c_n) = 1$. This point will lie inside of $\mathfrak{R}'_{n+1}$; therefore, the straight line $\eta_1 = c_1, \cdots, \eta_n = c_n$, passing through the point P, will have as its intersection with $\mathfrak{R}'_{n+1}$ some segment $\eta_1 = c_1, \eta_2 = c_2, \cdots, \eta_n = c_n$, $\eta' \leq \eta \leq \eta''$. Thus the inequality

$$\lambda(c_1, c_2, \cdots, c_n, \eta) \leq L$$

holds for those η and only those η which fall in the interval $< \eta', \eta'' >$. On the other hand according to proposition 2°, the inequality written above is a necessary and sufficient condition for the existence of a solution $f \subset E^*$ to system (5).

The second part of the assertion follows from the fact that every solution $f \subset E^*$ of system (5) satisfies the inequality $\lambda(c_1, \cdots, c_n, \eta) \leq \|f\| \leq L$.

This proposition may be generalized in the following manner:

5°. *If the elements* $x_1, x_2, \cdots, x_n; y_1, \cdots, y_p$ *are linearly independent and* $L > \lambda(c_1, \cdots, c_n)$, *then the points* $(\eta_1, \eta_2, \cdots, \eta_p)$ *for which the system*

$$\left.\begin{aligned} f(x_i) &= c_i \quad (i = 1, 2, \cdots, n), \\ f(y_k) &= \eta_k \quad (k = 1, 2, \cdots, p), \end{aligned} \right\} \quad \|f\| \leq L, \tag{6}$$

has some solution $f \subset E^*$, *fill a convex* p*-dimensional body.*

Without loss of generality we may consider that $y_k = x_{n+k}$ $(k = 1, 2, \cdots, p)$ and $\lambda(c_1, \cdots, c_n) = 1$ (and therefore $L > 1$). We reason further in a manner analogous to that of the previous proposition.

We shall consider a convex body $\mathfrak{R}'_{n+p}$, defined by the inequality

$$\lambda(\xi_1, \xi_2, \cdots, \xi_{n+p}) \leq L.$$

Obviously, $\mathfrak{R}'_{n+p} = L \cdot \mathfrak{R}_{n+p}$. By proposition 3° there exists a point $Q(c_1, c_2, \cdots, c_n, c^0_{n+1}, \cdots, c^0_{n+p})$ such that

$$\lambda(c_1, \cdots, c_n, c^0_{n+1}, \cdots, c^0_{n+p}) = \lambda(c_1, \cdots, c_n) = 1.$$

The point O lies inside $\mathfrak{R}'_{n+p}$; therefore, the linear manifold $\xi_1 = c_1, \cdots, \xi_n = c_n$ passing through the point O has an intersection with $\mathfrak{R}'_{n+p}$ which is some p-dimensional convex body $\mathfrak{R}$.

Proposition 5° follows from this result and proposition 2°.

6°. *If the space E is of dimension not less than $n+2$, then for any $L > \lambda(c_1, \cdots, c_n)$ the system*

$$f(x_i) = c_i \ (i = 1, 2, \cdots, n), \quad \| f \| = L \tag{7}$$

has at least a continuum of different solutions.

In fact, we shall add to the system $x_1, \cdots, x_n$ some system of elements $y_1, y_2, \cdots, y_p$ $(p \geq 2)$ to obtain the system of linearly independent elements $x_1, \cdots, x_m, y_1, \cdots, y_p$ and consider the convex p-dimensional body $\mathfrak{R}$ defined in proposition 5°.

To every point $(\eta_1, \cdots, \eta_p) \subset \mathfrak{R}$ there corresponds some solution $f \subset E^*$ of system (6) and here $\lambda(c_1, \cdots, c_n, \eta_1, \cdots, \eta_p) \leq \| f \| \leq L$. Therefore, to each boundary point $(\eta_1, \cdots, \eta_p)$ of the body $\mathfrak{R}$ there corresponds its own solution of system (7), for which $\| f \| = L$. Since the boundary of $\mathfrak{R}$ consists of a continuum of different points, proposition 6° follows.

We shall introduce the following definitions.

By an *extremal element* of a given linear functional f we shall mean any element $x \neq \theta$ for which

$$| f(x) | = \| f \| \cdot \| x \|. \tag{8}$$

The linear functional f will be called *normal* if it has extremal elements and they all differ from one another by a scalar multiple.

As we know, [1] for every element x there exists a functional $f \subset E^*$ having this element x as extremal.

We shall say that an element x is *normal* if the functional f is determined by equality (8) to within a scalar multiple.

It is not hard to see that

7°. *In order for the element $x = \Sigma_1^n \xi_i x_i$ $(\Sigma_1^n c_i \xi_i = 1)$ to be a minimizing*

1) Cf. Theorem 7 of the previous article.

element of problem B, *it is necessary and sufficient that the element* x *be an extremal element of any arbitrarily selected solution of system* (1) *of problem* A *with minimal norm, i.e., a solution of the system*

$$f(x_i) = c_i \ (i = 1, 2, \cdots, n), \quad \| f \| = \lambda(c_1, \cdots, c_n) > 0. \tag{9}$$

Indeed, let f_0 be some solution of system (9); if $y = \Sigma_1^n a_i x_i$ is some minimizing aggregate of problem B, i.e., $\Sigma_1^n a_i c_i = 1$ and $\| y \| = \frac{1}{\lambda(c_1, \cdots, c_n)}$, then

$$1 = \sum_1^n c_i a_i = f_0(y) = \| f_0 \| \ \| y \|, \tag{10}$$

or $\| f_0 \| = \lambda$.

Conversely, if $y = \Sigma_1^n a_i x_i$ ($\Sigma_1^n a_i c_i = 1$) is an extremal element of the functional f_0, then equality (10) holds and therefore $\| y \| = \frac{1}{\lambda}$.

8°. *If for any* ξ_i $(i = 1, 2, \cdots, n)$ *the element* $\Sigma_1^n \xi_i x_i$ *is normal,* [1] *then the system*

$$f(x_i) = c_i \ (i = 1, 2, \cdots, n), \quad \| f \| = \lambda(c_1, \cdots, c_n) > 0 \tag{11}$$

has one and only one solution $f \subset E^*$.

Indeed, let $y = \Sigma_1^n a_i x_i$ ($\Sigma_1^n c_i a_i = 1$) be some minimizing aggregate of problem B; then by 7°, every solution of system (11) having y as an extremal element is exactly defined by this system except for a multiplicative constant (since y is normal). But because $f(y) = \Sigma_1^n a_i c_i = 1$, this constant is uniquely determined.

In a manner analogous to 7° it follows that:

9°. *If system* (11) *has at least one normal solution then the minimizing element* $\Sigma_1^n \xi_i x_i$ ($\Sigma_1^n c_i \xi_i = 1$) *of problem* B *is determined uniquely.*

In addition,

10°. *If problem* B *has two different minimizing elements, then it has at least a continuum of them.*

In this connection, if $y_1 = \Sigma_1^n a_i' x_i$ and $y_2 = \Sigma_1^n a_i'' x_i$ are two minimizing elements of problem B, then for any $t (0 \le t \le 1)$ the element $y = t y_1 + (1 - t) y_2 = \Sigma_1^n a_i x_i$ is also a minimizing element of problem B, since $\Sigma_1^n a_i c_i = t (\Sigma_1^n a_i' c_i) + (1 - t) \Sigma_1^n a_i'' c_i = 1$ and $\| y \| \le t \| y_1 \| + (1 - t) \| y_2 \| = \lambda$, and therefore $\| y \| = \lambda$.

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1) This requirement may be replaced by a weaker one, namely: that even one minimizing element of problem B be normal.

Analogous remarks could be made in regard to system (11) or (7) for the case when this system has two linearly independent solutions f_1 and f_2.

In concluding this section we note [1] that in order for any functional $f \subset E^*$ either to have no extremal elements at all or for its extremal elements to differ from one another by only a scalar multiple (f is a normal functional), it is necessary and sufficient that the space E be strictly normed, i.e., that in the relation

$$\| x + y \| \leq \| x \| + \| y \|$$

the equality sign applies if and only if $y = \lambda x$ or $x = \lambda y$, where $\lambda \geq 0$.

Indeed, let

$$\| x_1 + x_2 \| = \| x_1 \| + \| x_2 \|$$

and elements x_1 and x_2 be linearly independent. Consider the functional $f(x)$, having the element $y = x_1 + x_2$ as an extremal one. For this functional

$$| f(x_1 + x_2) | = \| f \| \cdot \| x_1 + x_2 \| = \| f \| \cdot \| x_1 \| + \| f \| \cdot \| x_2 \|. \tag{12}$$

On the other hand,

$$| f(x_1 + x_2) | \leq | f(x_1) | + | f(x_2) | \leq \| f \| \cdot \| x_1 \| + \| f \| \cdot \| x_2 \|. \tag{13}$$

Combining (12) and (13) we find that

$$| f(x_1) | = \| f \| \, \| x_1 \| \text{ and } | f(x_2) | = \| f \| \, \| x_2 \|.$$

Thus, the condition that the space be strictly normed is a necessary condition that there not be two linearly independent extremal elements for any functional $f \subset E^*$.

This condition is also sufficient, because if

$$f(x_1) = \| f \| \, \| x_1 \| \text{ and } f(x_2) = \| f \| \, \| x_2 \|,$$

then

$$\| f \| \cdot (\| x_1 \| + \| x_2 \|) = f(x_1 + x_2) \leq \| f \| \, \| x_1 + x_2 \|,$$

from which

$$\| x_1 + x_2 \| = \| x_1 \| + \| x_2 \|.$$

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1) For this remark the considerations which follow, the author is indebted to T. M. Gel'fand and V. L. Šmul'yan.

We leave it to the reader to prove that the last equality is satisfied if and only if the segment connecting the points

$$e_1 = \frac{x_1}{\| x_1 \|} \text{ and } e_2 = \frac{x_2}{\| x_2 \|}$$

belongs to the boundary of the unit sphere $\| x \| \leq 1$.

Thus, the condition that a space E be strictly normed is equivalent to the condition that the unit sphere does not contain segments on its boundary.

§2

After the previous general considerations, we turn to some concrete examples of problems A and B.

1. We shall begin with elementary examples. Let M_m represent an m-dimensional space of vectors $x = (x^1, x^2, \cdots, x^m)$ [1] with the norm defined by

$$\| x \| = \max_{1 \leq i \leq m} | x^i |.$$

We introduce the unit vectors

$$e_1 = (1, 0, \ldots, 0)$$
$$\cdots\cdots\cdots$$
$$e_m = (0, 0, \ldots, 1).$$

Obviously,

$$x = \sum_1^m x^i e_i.$$

Now let $f(x)$ be some linear functional. Then

$$f(x) = \sum_1^m x^i f(e_i) = \sum_1^m f^i x^i,$$

where the numbers

$$f^i = f(e_i) \quad (i = 1, 2, \cdots, m).$$

Thus, every linear functional $f \subset M_m^*$ in the given case is defined by some vector $(f^1, f^2, \cdots, f^m)$, which obviously may have completely arbitrary coordi-

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1) The coordinates x^i $(i = 1, 2, \cdots, n)$ are of course real numbers.

nates. We shall identify the functional $f \subset M_m^*$ with the vector $(f^1, f^2, \cdots, f^m)$ and instead of $f(x)$ we shall write (f, x). Thus

$$f(x) = (f, x) = \sum_1^m f^i x^i.$$

We shall find the norm $\| f \|$ of the functional $f(x)$. Obviously

$$| (f, x) | \leq \sum_1^m | f^i | \cdot \max_{1 \leq i \leq m} | x^i | = \| x \| \cdot \sum_1^m | f^i |.$$

If in this relation we set $x^i = \operatorname{sign} f^i$ $(i = 1, 2, \cdots, m)$, then we will have equality. Therefore

$$\| f \| = \sup \frac{| (f, x) |}{\| x \|} = \sum_1^m | f^i |.$$

In this manner in the given case the conjugate space $L_m = M_m^*$ may be considered as the space of vectors $f = (f^1, f^2, \cdots, f^m)$ with the norm defined as

$$\| f \| = \sum_1^m | f^i |.$$

It is easy to see that the spaces M_m and L_m are mutually conjugate,[1] i.e., if we take the space L_m as basic then the conjugate space to it is the space M_m.

We shall show that the vector $f \subset L_m$ is normal if and only if all of the coordinates of the vector f are different from zero:

$$f^i \neq 0 \quad (i = 1, 2, \cdots, m).$$

In fact, with this condition satisfied it follows from the equality

$$| (f, x) | = | \sum_1^m f^i x^i | = \sum_1^m | f^i | \cdot \max_{1 \leq i \leq m} | x^i | = \| f \| \cdot \| x \|$$

that

$$x^i = C \operatorname{sign} f^i \quad (i = 1, 2, 3, \cdots, m). \tag{14}$$

If to the contrary some $f^k = 0$, then in the extremal element $x = (x^1, x^2, \cdots, x^m)$ of the functional f the corresponding coordinate x^k may be

---

1) In general, for every finite dimensional space E with any definition of the norm it is true that $(E^*)^* = E$.

chosen completely arbitrarily (independent of the other coordinates).

It is not hard to verify also that the vector $x = (x^1, x^2, \cdots, x^m)$ is normal if and only if all of its coordinates equal zero with the exception of one of them.

2. Now let

$$\begin{aligned} x_1 &= (a_1^1, \cdots, a_1^m), \\ &\cdots\cdots\cdots\cdots \qquad (n < m) \\ x_n &= (a_n^1, \cdots, a_n^m) \end{aligned}$$

be any system of n linearly independent elements of M_m. If we try to formulate problem B, then we come to the following problem:

I. *Given a matrix of rank* $n(\leq m)$

$$\left\| \begin{matrix} a_1^1 \cdots a_n^1 \\ a_1^2 \cdots a_n^2 \\ \cdots\cdots\cdots \\ a_1^m \cdots a_n^m \end{matrix} \right\|$$

and numbers $c_1, c_2, \cdots, c_n$ $(\Sigma_1^n c_i^2 > 0)$.

Find

$$\frac{1}{\lambda} = \min_{\xi} \max_{1 \leq i \leq m} | a_1^i \xi_1 + a_2^i \xi_2 + \cdots + a_n^i \xi_n | \tag{15}$$

with the additional condition regarding the ξ *that*

$$\sum_1^n c_i \xi_i = 1. \tag{15_1}$$

We shall now formulate the result to which the basic proposition 2° §1 reduces.

Ia. *The desired number* λ *of problem* I *may also be defined by the equality*

$$\lambda = \min (| f^1 | + | f^2 | + \cdots + | f^m |) \tag{16}$$

with the supplementary conditions on $f^i (i = 1, 2, \cdots, m)$:

$$\begin{aligned} &a_1^1 f^1 + a_1^2 f^2 + \cdots + a_1^m f^m = c_1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_n^1 f^1 + a_n^2 f^2 + \cdots + a_n^m f^m = c_n. \end{aligned} \tag{16_1}$$

In particular, if $n = m$, *then*

$$\lambda = \sum_1^m |f^i|,$$

where the numbers f^i $(i = 1, 2, \cdots, m)$ *are uniquely determined by* (16_1).

Now let the minimum in (16) be attained for some vector $f = (f^1, f^2, \cdots, f^m)$, all of whose coordinates are different from zero. Then in agreement with proposition 9° the vector $\delta = (\delta^1, \delta^2, \cdots, \delta^m)$ with coordinates

$$\delta^i = a_1^i \xi_1 + \cdots + a_n^i \xi_n \quad (i = 1, 2, \cdots, m),$$

for which (15) is attained is uniquely determined (as the extremal element f). That is, according to (14) we have

$$\delta^i = C \operatorname{sign} f^i \quad (i = 1, 2, \cdots, m);$$

and, moreover, by virtue of the fact that

$$(f, \delta) = \left(f, \sum_1^n \xi_i x_i\right) = \sum_1^n c_i \xi_i = 1,$$

we have

$$C = \frac{1}{\sum_1^m |f^i|} = \lambda.$$

Thus we arrive at the proposition:

Ib. *If the minimum in* (16) *is attained for some vector* $f_0 = (f_0^1, f_0^2, \cdots, f_0^m)$, *all of whose coordinates are different from zero, then*

$$\min_{\xi} \max_{1 \le i \le m} |a_1^i \xi_1 + \cdots + a_n^i \xi_n| = \frac{1}{\sum_1^m |f_0^i|} \quad \left(\sum_1^n c_i \xi_i = 1\right).$$

The minimax is attained here at those points ξ_i $(i = 1, 2, \cdots, n)$ *and only those points for which also*

$$a_1^i \xi_1 + \cdots + a_n^i \xi_n = \frac{\operatorname{sign} f_0^i}{\sum_1^m |f_0^i|} \quad (i = 1, 2, \cdots, m). \tag{17}$$

We note that the condition stated (the existence of an f_0 with all coordinates

different from zero) is easily verified for $n = m$. In this case the numbers ξ_i $(i = 1, 2, \cdots, m)$, for which the minimax is attained are uniquely determined by (17).

3. Problems I and the propositions Ia and Ib have an immediate relation to the theory of "best approximate" solutions to incompatible systems of equations.

Let there be given an incompatible system of equations

$$a_1^i \xi_1 + \cdots + a_{n-1}^i \xi_{n-1} = l^i \quad (i = 1, 2, \cdots, m), \tag{18}$$

in which the number of equations is greater than the number of unknowns $(m \geq n)$.

In this case the question of solving system (18) exactly is of course eliminated; still, we may raise the question of the best approximate solution of system (18) in some sense, for example in Čebyšev's sense.

By the latter we mean the determination of values ξ_i $(i = 1, 2, \cdots, n-1)$, for which the maximal deviation

$$\max_{1 \leq i \leq m} | a_1^i \xi_1 + \cdots + a_{n-1}^i \xi_{n-1} - l^i |$$

has the smallest possible value.

Since we can of course consider the vectors $(a_i^1, a_i^2, \cdots, a_i^m)$ $(i = 1, 2, \cdots, n-1)$ and the vector $(l^1, l^2, \cdots, l^m)$ to be linearly independent without loss of generality here, this problem is a special case (to which we may reduce the general case) of problem I, if in the latter we set

$$c_1 = c_2 = \cdots = c_{n-1} = 0, \; c_n = 1, \; a_n^i = -l^i \quad (i = 1, 2, \cdots, m).$$

Applying propositions Ia and Ib to this case we obtain the following deductions:

Ic. *If the rank of the matrix*

$$\begin{Vmatrix} a_1^1 & \cdots & a_{n-1}^1 & l^1 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_1^m & \cdots & a_{n-1}^m & l^m \end{Vmatrix}$$

equals $n (n \leq m)$, *then*

$$\min_{\xi} \max_{1 \leq i \leq m} | a_1^i \xi_1 + a_2^i \xi_2 + \cdots + a_{n-1}^i \xi_{n-1} - l^i | = \frac{1}{\lambda}, \tag{19}$$

where

$$\lambda = \min \sum_{1}^{m} |f^i| \tag{20}$$

with the additional conditions on the f^i $(i = 1, 2, \cdots, m)$ *that*

$$\left.\begin{aligned} &a_1^1 f^1 + a_1^2 f^2 + \cdots + a_1^m f^m = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{n-1}^1 f^1 + a_{n-1}^2 f^2 + \cdots + a_{n-1}^m f^m = 0 \end{aligned}\right\} \tag{20_1}$$

and

$$l^1 f^1 + \cdots + l^m f^m = -1. \tag{20_2}$$

If the minimum in (20) *is attained for some* $f_0 = (f_0^1, f_0^2, \cdots, f_0^m)$, *which has all of its coordinates different from zero, then the minimax in* (19) *is attained for those* $\xi_1, \xi_2, \cdots, \xi_{n-1}$ *and only those for which simultaneously*

$$a_1^i \xi_1 + \cdots + a_{n-1}^i \xi_{n-1} - l^i = \frac{\operatorname{sign} f^i}{\lambda} \quad (i = 1, 2, \cdots, m).$$

For $n = m$ formula (20) follows by substitution of the formula:

$$\lambda = \sum_{1}^{m} |f^i|,$$

where f^i $(i = 1, 2, \cdots, m)$ are defined by systems (20_1) and (20_2).

For this particular case under the assumption that all $f^i \neq 0$ $(i = 1, 2, \cdots, m)$ (which, as we see, is inessential for formula (19)) proposition Ic was proved by E. Remez [1] [16].

The question may be asked: *what conditions must be satisfied by the matrix*

$$\left\| \begin{matrix} a_1^1 & \cdots & a_{n-1}^1 \\ \cdots & \cdots & \cdots \\ a_1^m & \cdots & a_{n-1}^m \end{matrix} \right\|, \tag{21}$$

in order that for any l^i $(i = 1, 2, \cdots, m)$ *the minimax in* (19) *is attained for a unique set of values*

$$\xi_1, \xi_2, \cdots, \xi_{n-1}.$$

---

1) Cf. the problem on page 131 (Chapter VIII) of the work cited. In this work one finds a series of other interesting conditions on incompatible systems. All of them, among others, may be quite simply obtained by Haar's [28] method.

The answer to this question can be obtained immediately from the solution found by Haar [28] of a more general question into which we can not go here, unfortunately, because of lack of space. We shall only note that by virtue of Haar's result the condition sought is included in the condition that all minors of order $(n-1)$ of the determinant (21) are non-zero. [1]

We may also demonstrate that the minimax in (19) preserves its original value if of the linear forms

$$a_1^i \xi_1 + a_2^i \xi_2 + \cdots + a_{n-1}^i \xi_{n-1} - l^i \quad (i = 1, 2, \cdots, m) \tag{22}$$

under the minimax sign in (19) we leave only n of the forms [2] chosen by some means.

Thus, solving the problem of determining the corresponding minimax for every one of the $\binom{m}{n}$ combinations of n forms (22), by selecting the largest of the values determined (a finite number of operations) we may determine the exact value of the minimax (19).

4. We now interchange the roles of the spaces M_m and L_m. Then problem B has the following form:

I'. *Given a matrix of rank* n

$$\left\| \begin{array}{ccc} f_1^1 & \cdots & f_n^1 \\ \cdots & \cdots & \cdots \\ f_1^m & \cdots & f_n^m \end{array} \right\|$$

and numbers $c_1, c_2, \cdots, c_n$ $(\sum_1^n c_i^2 > 0)$.

Find

$$\min \sum_{i=1}^{m} | \xi_1 f_1^i + \cdots + \xi_n f_n^i | = \frac{1}{\mu}$$

---

1) This result is also true if an infinite number of linear forms figure in (19) $(m = \infty)$. Details about Haar's problem may be found in a book by N. Ahiezer [2b].

2) Cf. E. Remez [16]; there one finds a generalization of this fact for the case of an infinite number of forms (22). All of these results may be quite simply obtained by Haar's method if we take into consideration the properties of the smallest convex set containing the given set. At the time when the proofs were being read, the present author was introduced to an article by L. G. Šnirel'man (Proceedings of the Academy of Sciences U.S.S.R., No. 1, 1938) which appeared recently. In this article a new very simple and ingenious method for establishing a similar kind of proposition is adduced; it still remains to show that the concrete results, which L. T. Šnirel'man obtained by his method, were previously established by Haar [28] and Remez [16] in another way.

with the additional condition on the ξ:

$$\sum_1^n c_i \xi_i = 1.$$

The basic proposition 2° §1 in this case gives:

I'a. *The number* μ *sought in problem* I' *may also be determined by the equality*

$$\mu = \min_x \max_{1 \le i \le m} |x^i|$$

with the additional condition on the x^i $(i = 1, 2, \cdots, m)$ *that*

$$\left.\begin{aligned} f_1^1 x^1 + f_1^2 x^2 + \cdots + f_1^m x^m &= c_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ f_n^1 x^1 + f_n^2 x^2 + \cdots + f_n^m x^m &= c_m. \end{aligned}\right\} \tag{23}$$

In particular, if $n = m$ *then* $\mu = \max_{1 \le i \le m} |x^i|$,

where the numbers x^i $(i = 1, 2, \cdots, m)$ *are uniquely determined from* (23).

The analogue of proposition Ib is uninteresting and we shall omit it.

We shall now consider the incompatible system of equations

$$f_1^i \xi_1 + \cdots + f_{n-1}^i \xi_{n-1} = h^i \quad (i = 1, 2, \cdots, m;\ m \ge n).$$

We shall also seek ξ_i $(i = 1, 2, \cdots, n-1)$, so that the quantity

$$\sum_{i=1}^m |f_1^i \xi_1 + \cdots + f_{n-1}^i \xi_{n-1} - h^i|$$

is as small as possible. As before we shall consider the vectors $(f_i^1, \cdots, f_i^m)$ $(i = 1, 2, \cdots, n-1)$ and the vector $(h^1, \cdots, h^m)$ to be linearly independent. For this problem (as a particular case of problem I') we may formulate such an analogue to proposition Ic.

I'c. *If the rank of the matrix*

$$\left\| \begin{matrix} f_1^1 \cdots f_{n-1}^1 h^1 \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ f_1^m \cdots f_{n-1}^m h^m \end{matrix} \right\|$$

equals n $(n \le m)$, *then*

$$\min \sum_{i=1}^{m} | f_1^i \xi_1 + \cdots + f_{n-1}^i \xi_{n-1} - h^i | = \frac{1}{\mu},$$

where

$$\mu = \min_{x} \max_{1 \le i \le m} | x^i | \tag{24}$$

with the additional conditions on the $x^i (i = 1, 2, \cdots, m)$ *that*

$$\left.\begin{aligned} & f_1^1 x^1 + \cdots + f_1^m x^m = 0, \\ & \cdots\cdots\cdots\cdots\cdots \\ & f_{n-1}^1 x^1 + \cdots + f_{n-1}^m x^m = 0 \end{aligned}\right\} \tag{24_1}$$

and the condition

$$h^1 x^1 + \cdots + h^m x^m = 1. \tag{24_2}$$

For $n = m$, the vector $x = (x^1, x^2, \cdots, x^m)$ is uniquely defined by (24_1) and (24_2) and it follows that formula (24) can be replaced by the simpler formula

$$\mu = \max_{1 \le i \le m} | x^i |.$$

We note that problems I and I′ are not only reciprocal, so to speak, but that each one may be reduced to the other. We shall explain for example, how to reduce problem I to problem I′.

By proposition Ia finding a minimax (15) reduces to finding a minimum (16). But the general solution of system (16_1) may always be represented in the form

$$f^i = h^i - \sum_{k=1}^{p} \xi_k f_k^i \quad (p = m - n),$$

where $(h^1, h^2, \cdots, h^m)$ is some particular solution of system (16_1) and $(f_k^1, \cdots, f_k^m)$ $(k = 1, 2, \cdots, p)$ are p linearly independent solutions to the corresponding homogeneous problem.

Therefore the minimum λ of (16) may also be determined as

$$\lambda = \min_{\xi} \sum_{i=1}^{m} | h^i - \sum_{k=1}^{p} \xi_k f_k^i |,$$

which proves our assertion.

§3

In this section we shall consider applications of the general results of §1 to some functional spaces.

Let $\mathfrak{M}$ be some bounded measurable set on the real axis $t(-\infty < t < \infty)$. We shall denote by $E_{\mathfrak{M}}$ the linear space of all functions $x(t)$ measurable and summable on $\mathfrak{M}$ with the norm defined [1] as

$$\| x \| = \int_{\mathfrak{M}} | x(t) | \, dt.$$

We shall consider as known [2] that every linear functional $f \subset E^*_{\mathfrak{M}}$ has the following analytic expression:

$$f(x) = \int_{\mathfrak{M}} x(t)\, \alpha(t)\, dt, \tag{25}$$

where $\alpha(t)$ is some function on $\mathfrak{M}$ measurable and essentially bounded; moreover, the norm of the functional f is computed by the formula:

$$\| f \| = \operatorname{vrai\,max}_{t \subset \mathfrak{M}} | \alpha(t) |. \tag{25_1}$$

By virtue of (25) and (25_1) the condition for the extremality of the function $x(t) \subset E_{\mathfrak{M}}$ relative to the functional f can be written in the form of the equality

$$\int_{\mathfrak{M}} x(t)\, \alpha(t)\, dt = \pm \int_{\mathfrak{M}} | x(t) | \, dt \cdot \operatorname{vrai\,max}_{t \subset \mathfrak{M}} | \alpha(t) |.$$

But it is easy to see that this equality holds if and only if

$$\alpha(t) = \epsilon \operatorname{vrai\,max}_{t \subset \mathfrak{M}} | \alpha(t) | \operatorname{sign} x(t) \text{ almost everywhere on } \mathfrak{N}_x, \tag{26}$$

where $\mathfrak{N}_x$ denotes the set of points $t \subset \mathfrak{M}$, for which $x(t) \neq 0$, and $\epsilon = \pm 1$.

Therefore, the function $x(t)$ is normal (like the element $E_{\mathfrak{M}}$) if and only if $\mathfrak{M} - \mathfrak{N}_x$ is a set of measure zero, since in this case and only in this case the

---

1) Thus we shall not distinguish two functions $x(t) \subset E_{\mathfrak{M}}$ and $y(t) \subset E_{\mathfrak{M}}$, if they coincide almost everywhere on $\mathfrak{M}$ since in this case $\| x - y \| = 0$. This must also be kept clearly in mind when speaking about linearly independent functions $x_i(t)$ $(i = 1, 2, \cdots, n)$.

2) Cf. page 65 of Banach's book; the deduction of formula (25) is mentioned there for the case $\mathfrak{M} = \langle a, b \rangle$, but the reasoning is easily generalized to the general case.

function $\alpha(t)$ is exactly determined from (26) except for a scalar multiple and is in fact given by the formula:

$$\alpha(t) = C \operatorname{sign} x(t) \text{ (almost everywhere on } \mathfrak{M}).$$

In the space $E_{\mathfrak{M}}$ problems A and B will take the following form:

A'. *Given n functions $x_i(t) \subset E_{\mathfrak{M}}$ $(i = 1, 2, \cdots, n)$. Find necessary and sufficient conditions on the numbers $c_1, c_2, \cdots, c_n$, L $(\Sigma_1^n c_i^2 > 0,\ L > 0)$, for the existence of a measurable function $\alpha(t)$ for which:*

$$\int_{\mathfrak{M}} x_i(t)\,\alpha(t)\,dt = c_i \quad (i = 1, 2, \cdots, n), \quad \sup |\,\alpha(t)\,| \leq L.$$

B'. *Given n linearly independent functions $x_i(t) \subset E_{\mathfrak{M}}$ and n numbers $c_1, c_2, \cdots, c_n$ $(\Sigma_1^n c_i^2 > 0)$. Find the minimum*

$$\frac{1}{\lambda} = \min_{\sum_1^n c_i \xi_i = 1} \int_{\mathfrak{M}} |\,\xi_1 x_1(t) + \cdots + \xi_n x_n(t)\,|\,dt.$$

In examining this problem we shall first show that:

1°. If $y = \Sigma_1^n a_i x_i(t)$ and $z = \Sigma_1^n b_i x_i(t)$ are two minimizing elements of problem B, *then the equality*

$$\operatorname{sign}\left(\sum_1^n a_i x_i(t)\right) = \operatorname{sign}\left(\sum_1^n b_i x_i(t)\right) \tag{27}$$

holds almost everywhere in the intersection of the sets $\mathfrak{N}_y$ and $\mathfrak{N}_z$.

In fact, let $\alpha(t)$ be some solution of problem A' for $L = \lambda(c_1, c_2, \cdots, c_n)$. Then according to proposition 7° §1, the functions $y(t)$ and $z(t)$ are extremal functions of the functional f_0 corresponding to function $\alpha(t)$ and therefore by virtue of (26) and the fact that $f_0(y) = f_0(z) = 1$

$$\alpha(t) = \lambda \operatorname{sign}\left[\sum_1^n a_i x_i(t)\right] \text{ (almost everywhere on } \mathfrak{N}_y),$$

$$\alpha(t) = \lambda \operatorname{sign}\left[\sum_1^n b_i x_i(t)\right] \text{ (almost everywhere on } \mathfrak{N}_z).$$

From this (27) follows immediately.

We shall call n functions $x_i(t) \subset E_{\mathfrak{M}}$ $(i = 1, 2, \cdots, n)$ *completely independent* if any of their linear combinations $\Sigma_1^n \xi_i x_i(t)$ $(\Sigma_1^n \xi_i^2 > 0)$ gives a normal function,

i.e., does not vanish on any set of measure > 0.

After all that has been said, it is not difficult to conclude from propositions 6°, 7°, 8° §1 that

Theorem 1. *Let* $x_i(t)$ $(i = 1, 2, \cdots, n)$ *be completely independent. If* $y(t) = \Sigma_1^n a_i x_i(t)$ $(\Sigma_1^n c_i a_i = 1)$ *is a minimizing element of problem* B′, *then any other function* $z(t) = \Sigma_1^n b_i x_i(t)$ $(\Sigma_1^n c_i b_i = 1)$ *will also be a minimizing element of problem* B′ *if and only if*

$$\operatorname{sign}\Big[\sum_1^n b_i x_i(t)\Big] = \operatorname{sign}\Big[\sum_1^n a_i x_i(t)\Big] \text{ (almost everywhere on } \mathfrak{M}).$$

Theorem 2. *Let the functions* $x_i(t)$ $(i = 1, 2, \cdots, n)$ *be completely independent.*

In order for the system

$$\int_{\mathfrak{M}} x_i(t)\,\alpha(t)\,dt = c_i \ (i = 1, 2, \cdots, n), \ \sup|\alpha(t)| \le L \tag{28}$$

to have a continuum of essentially different solutions $\alpha(t)$ *it is necessary and sufficient that*

$$\frac{1}{L} < \frac{1}{\lambda} = \min_{\sum_1^n c_i \xi_i = 1} \int_{\mathfrak{M}} |\xi_1 x_1(t) + \cdots + \xi_n x_n(t)|\,dt. \tag{29}$$

In order for system (28) *to have essentially one solution and only one it is necessary and sufficient that* $L = \lambda$.

With this condition satisfied the solution $\alpha(t)$ *is determined by the formula*

$$\alpha(t) = L \operatorname{sign}\Big[\sum_1^n a_i x_i(t)\Big], \text{ (almost everywhere on } \mathfrak{M}),$$

where the aggregate $\Sigma_1^n a_i x_i(t)$ $(\Sigma_1^n a_i c_i = 1)$ *is any minimizing element of problem* B, *i.e., an element for which the minimum is attained in* (29).

We shall clarify the role of Theorems 1 and 2 and their significance in several particular examples.

For example, let us raise the question of determining

$$\min_{\xi} \int_{-1}^{1} |\xi_0 + \xi_1 t + \cdots + \xi_n t^n|\,dt \tag{30}$$

with the supplementary condition that

$$\sum_0^n c_i \xi_i = 1. \qquad (30_1)$$

In this particular problem B′:

$$\mathfrak{M} = \langle -1,\ 1 \rangle,\ x_i(t) = t^i \quad (i = 0,\ 1, \cdots,\ n).^{1)}$$

Since the powers t^i are completely independent, we may apply Theorem 1. Let

$$P(t) = A_0 + A_1 t + \cdots + A_n t^n \quad (\sum_0^n c_i A_i = 1)$$

be some polynomial for which minimum (30) is attained.

Let $\alpha_1 < \alpha_2 < \cdots < \alpha_p (p \leq n)$ be those points at which it changes sign. Let

$$Q(t) = C \prod_1^p (t - \alpha_i) = B_0 + B_1 t + \cdots + B_p t^p$$

and select the constant C so that

$$\operatorname{sign} P(t) = \operatorname{sign} Q(t)$$

and so that

$$\sum_0^p c_i B_i = 1.$$

By Theorem 1 the polynomial $Q(t)$ will be one of the desired minimizing polynomials. Hence, *the minimum* (30) *is always attained for some polynomial with simple real roots lying in the interval* $(-1, 1)$.

Moreover, by Theorem 1 a minimum is attained in (30) if and only if

$$\sum_0^n \xi_i t^i = G(t) P(t),$$

where $G(t)$ is an arbitrary polynomial non-negative in the interval $(-1, 1)$ subject to the one condition that equality (30_1) must be satisfied.

And so with the help of Theorem 1 we can very quickly clarify the nature of the solutions to the problem of determining the minimum (30). This problem was already the subject of the investigation on pages 98–104 of Article I.

What does Theorem 2 yield? Obviously it provides a connection of the problem just worked out with the problem which we called the L-problem in Chapter III of Article I.

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1) And therefore n has been substituted for $n + 1$.

Knowing qualitatively the nature of solutions to these problems one may show without much difficulty that the number $\lambda = \lambda(c_1, c_2, \cdots, c_n)$ is a root of the algebraic equation mentioned in §4, Chapter III, Article I. We shall not, as we might, present a proof that λ is the largest root of this equation without the methods of functions of a complex variable developed in Article I.

Analogous remarks may be made about the corresponding trigonometric problems which arise if we set $\mathfrak{M} = <0, 2\pi>$, $x_0(t) = 1$, $x_1(t) = \cos t$, $x_2(t) = \sin t$, etc.

2. We shall now consider the problem of determining the minimum of the integral

$$\frac{1}{\lambda} = \min_{\xi} \int_{\mathfrak{M}} | g(t) - \sum_{1}^{n} \xi_i x_i(t) | \, dt, \tag{31}$$

where $x_i(t)$ $(i = 1, 2, \cdots, n)$ and $g(t)$ are given functions from $E_{\mathfrak{M}}$. Setting $x_{n+1}(t) = g(t)$ we may consider this problem as a particular case of problem B′ in which n is substituted for $n + 1$, $c_1 = \cdots = c_n = 0$, $c_{n+1} = 1$.

If the minimum in (31) is attained for some $\xi_i = a_i$ $(i = 1, 2, \cdots, n)$, then the function $\Sigma_1^n a_i x_i(t)$ will be called the *least deviation aggregate* of the function $g(t)$ in the sense (L).

We shall show that:

2°. *For any functions* $x_i(t) \subset E_{\mathfrak{M}}$ $(i = 1, 2, \cdots, n)$ *a function* $g(t) \subset E_{\mathfrak{M}}$ *can always be found for which the aggregate* $\Sigma_1^n \xi_i x_i(t)$ *which deviates least from it in the sense of* (L) *is not uniquely determined.*

Indeed, we shall first select a function $g_1(t)$ such that for any coefficients ξ_i $(i = 1, 2, \cdots, n)$ the function

$$h(t) = g_1(t) - \sum_{1}^{n} \xi_i x_i(t)$$

is normal, i.e., vanishes only on a set of measure zero.

This is always possible since, for example, an $\alpha (0 \leq \alpha \leq 1)$ can always be found so that the function $g_1(t) = (t - t_0)^{\alpha}$ for $t_0 < \inf t$ $(t \subset \mathfrak{M})$ will satisfy this condition. [1)]

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1) In fact, if such an α could not be found, then to every $\alpha (0 \leq \alpha \leq 1)$ there would correspond numbers $\xi_i^{(\alpha)}$ $(i = 1, 2, \cdots, n)$ such that

Let

$$\frac{1}{\lambda_1} = \min_{\xi} \int_{\mathfrak{M}} \left| g_1(t) - \sum_1^n \xi_i x_i(t) \right| dt$$

and let

$$\frac{1}{\lambda_1} = \int_{\mathfrak{M}} \left| g_1(t) - \sum_1^n a_i x_i(t) \right| dt.$$

We shall now consider the system of equations

$$\int_{\mathfrak{M}} \alpha(t)\, x_i(t)\, dt = 0 \ (i = 1, 2, \cdots, n), \quad \int_{\mathfrak{M}} \alpha(t)\, g(t)\, dt = 1, \quad \sup_{t \subset \mathfrak{M}} |\alpha(t)| = \lambda_1.$$

Since the element $h(t) = g_1(t) - \Sigma_1^n a_i x_i(t)$ is normal, the above system has the unique solution

$$\alpha(t) = \lambda_1 \operatorname{sign}\{ g_1(t) - \sum_1^n a_i x_i(t) \} = \lambda_1 \epsilon(t).$$

Thus we are convinced that there exists a function $\epsilon(t)$, equal to ± 1 almost

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$$(t - t_0)^{\alpha} - \sum_1^n \xi_i^{(\alpha)} x_i(t) = 0 \text{ for } t \subset \mathfrak{M}_{\alpha},$$

where $\mathfrak{M}_{\alpha} \subset \mathfrak{M}$ is some set of positive measure. Now let $\{\epsilon_n\}$ be a sequence of positive numbers which tend to zero ($\epsilon_n \to 0$). Since α runs through a continuum of values then for some $\epsilon' = \epsilon_n$ we may find a non-denumerable set of the collection $\mathfrak{M}_{\alpha}$ such that $\operatorname{mes} \mathfrak{M}_{\alpha} \geq \epsilon'$.

We shall consider non-empty intersections $\mathfrak{M}_{\alpha} \mathfrak{M}_{\beta}$ of pairs of these sets. It is obvious there is again a non-denumerable number of them. Therefore, there will exist an $\epsilon'' > 0$ such that a non-denumerable set of intersections $\mathfrak{M}_{\alpha} \mathfrak{M}_{\beta}$ will have measure larger than ϵ''. Extending such reasoning we prove that there exists some set $\mathfrak{M}_0 = \mathfrak{M}_{\alpha_1} \cdots \cdots \mathfrak{M}_{\alpha_{n+1}}$ (and even a non-denumerable number of such sets) with measure > 0. But then

$$(t - t_0)^{\alpha_j} = \sum_1^n \xi_i^{(\alpha_j)} x_i(t) \text{ for } t \subset \mathfrak{M}_0 \quad (j = 1, 2, \cdots, n+1),$$

and therefore there will exist $A_1, \cdots, A_{n+1}$ $(\Sigma A_i^2 > 0)$, such that

$$\sum_1^{n+1} A_j (t - t_0)^{\alpha_j} = 0 \text{ for } t \subset \mathfrak{M}_0;$$

and since $\operatorname{mes} \mathfrak{M}_0 > 0$ this is impossible since the left side of the equality is an analytic function (and more than that it has not more than n zeros).

The author is indebted to a discussion with B. Ya. Levin for this argument.

everywhere and such that

$$\int_{\mathfrak{M}} x_i(t)\,\epsilon(t)\,dt = 0 \quad (i = 1, 2, \cdots, n). \tag{32}$$

Each such function $\epsilon(t)$ allows us to construct the functions $g(t)$ whose existence is asserted in 2°. For this one needs only to set

$$g(t) = \epsilon(t)\left|\sum_1^n d_i x_i(t)\right| \quad \left(\sum_1^n d_i^2 > 0\right), \tag{33}$$

where $d_i\,(i = 1, 2, \cdots, n)$ are arbitrary numbers. In fact, for any ξ_i by (32), (33)

$$\int_{\mathfrak{M}}\left|g(t) - \sum_1^n \xi_i x_i(t)\right| dt \geq \int_{\mathfrak{M}}\left\{g(t) - \sum_1^n \xi_i x_i(t)\right\}\epsilon(t)\,dt =$$

$$= \int_{\mathfrak{M}} g(t)\,\epsilon(t)\,dt = \int_{\mathfrak{M}} |g(t)|\,dt.$$

On the other hand, for any $\theta(0 < \theta < 1)$ by (33) and (34)

$$\operatorname{sign}\{g(t) - \theta\sum_1^n d_i x_i(t)\} = \epsilon(t),$$

and therefore,

$$\int_{\mathfrak{M}}\left|g(t) - \theta\sum_1^n d_i x_i(t)\right| dt = \int_{\mathfrak{M}}\left\{g(t) - \theta\sum_1^n d_i x_i(t)\right\}\epsilon(t)\,dt =$$

$$= \int_{\mathfrak{M}} g(t)\,\epsilon(t)\,dt = \int_{\mathfrak{M}} |g(t)|\,dt,$$

i.e., for any $\theta(0 < \theta < 1)$ the aggregate $\theta\,\Sigma_1^n d_i x_i(t)$ deviates least from $g(t)$. Proposition 2° is proved.

In this connection of special interest is the circumstance that for $\mathfrak{M} = \langle a, b\rangle$ we may always construct continuous functions $x_i(t)$ $(i = 1, 2, \cdots, n)$, such that to any continuous function $g(t)$ there will correspond a unique aggregate $\Sigma_1^n \xi_i x_i(t)$ which deviates least from it in the sense (L). We shall now show this.

Let $\mathfrak{M} = \langle a, b\rangle$. We shall agree to say that the continuous functions $x_i(t) \subset E_{(a, b)}$ $(i = 1, 2, \cdots, n)$ form a system T if for any ξ_i $(i = 1, 2, \cdots, n)$

$(\Sigma_1^n \xi_i^2 > 0)$ the linear combination $\Sigma_1^n \xi_i x_i(t)$ has not more than $n-1$ zeros inside the interval $<a, b>$.

Obviously, for functions $x_i(t)$ $(i = 1, 2, \cdots, n)$ to form a system T it is necessary and sufficient that for all entirely distinct sets $t_1, t_2, \cdots, t_n$:

$$\Delta\begin{pmatrix} x_1, \cdots, x_n \\ t_1, \cdots, t_n \end{pmatrix} = \left| x_i(t_k) \right|_1^n \neq 0.$$

By the continuity of the functions $x_i(t)$ $(i = 1, 2, \cdots, n)$, the last condition is equivalent to the determinant Δ not changing its sign for $a < t_1 < t_2 < \cdots < t_n < b$.

And therefore, if the functions $x_i(t)$ $(i = 1, 2, \cdots, n)$ form a system T, then for any $a < t_1 < t_2 < \cdots < t_n < b$ the function

$$\Phi(t) = C\Delta\begin{pmatrix} x_1 x_2 \cdots x_{n-1} x_n \\ t_1 t_2 \cdots t_{n-1} t \end{pmatrix} \quad (C \neq 0)$$

changes sign when t passes through $t_i (i = 1, 2, \cdots, n)$ and has no zeros besides t_i $(i = 1, 2, \cdots, n-1)$. We shall use this for the proof of the following lemma:

Lemma. *If the continuous functions $x_i(t) \subset E_{(a, b)}$ $(i = 1, 2, \cdots, n)$ form a system T then whatever the numbers $t_1 < t_2 < \cdots < t_m$ $(m < n)$ lying inside (a, b), there always exists an aggregate $\Sigma_1^n A_i x_i(t)$ having the property that the aggregate changes sign at those points t_i $(i = 1, 2, \cdots, m)$ and only at those points.*

Proof. Let $n - m = 2p + 1$ be an odd number. We select the points $\tau_1, \tau_2, \cdots, \tau_p$ so that $t_m < \tau_1 < \tau_2 < \cdots < \tau_p < b$. Supposing that for $\epsilon > 0$ small enough so that $\tau_k + \epsilon < \tau_k$ $(k = 1, \cdots, p)$ and $\tau_p + \epsilon < b$, we construct a function

$$\Phi_\epsilon(t) = C_\epsilon \Delta\begin{pmatrix} x_1 \cdots x_m x_{m+1} x_{m+2} \cdots x_{n-1} x_n \\ t_1 \cdots t_m \tau_1 \quad \tau_1 + \epsilon \cdots \tau_p + \epsilon \quad t \end{pmatrix} = \sum_{i=1}^{n} A_i(\epsilon) x_i(t),$$

where $C_\epsilon > 0$ is chosen subject to the condition $\Sigma_{i=1}^n A_i^2(\epsilon) = 1$. Since $|A_i(\epsilon)| \leq 1$ $(i = 1, 2, \cdots, n)$, it is always possible to find a sequence $\epsilon_n \to 0$ such that there will simultaneously exist the limits $\lim_{n \to \infty} A_k(\epsilon_n)$ $(k = 1, 2, \cdots, n)$, which we may denote by A_k $(k = 1, 2, \cdots, n)$. Then the sequence of functions $\Phi_\epsilon(t)$ converges uniformly for $n \to \infty$ to the function $\Phi(t) = \Sigma_{i=1}^n A_i x_i(t)$. Since the function $\Phi_\epsilon(t)$ changes sign when t passes through the points $t_1 < t_2 < \cdots$

$\cdots < t_m < \tau_1 < \tau_1 + \epsilon < \tau_2 < \cdots < \tau_p < \tau_p + \epsilon$ and only at these points, it is easy to conclude that the limit function $\Phi(t)$ changes its sign when t passes through the points $t_1 < t_2 < \cdots < t_m$ and only for these points.

We shall now consider the case when $n - m = 2p$, an even number. In this case we may choose points $\tau_1, \tau_2, \cdots, \tau_p$ so that $t_m < \tau_1 < \tau_2 < \cdots < \tau_p = b$. Again supposing $\epsilon > 0$ is small enough so that $\tau_k + \epsilon < \tau_{k+1}$ $(k = 1, 2, \cdots, p-1)$, we construct a function

$$\Phi_\epsilon(t) = C_\epsilon \Delta \begin{pmatrix} x_1 \cdots x_m & x_{m+1} & x_{m+2} \cdots x_{n-1} & x_n \\ t_1 \cdots t_m & \tau_1 & \tau_1 + \epsilon \cdots \tau_{p-1} + \epsilon & b \end{pmatrix} = \sum_1^n A_i(\epsilon)\, x_i(t),$$

selecting $C_\epsilon > 0$ so that $\Sigma_1^n A_i^2(\epsilon) = 1$. We complete the argument as in the previous case.

We shall now prove a theorem which is essentially due to Jackson [31]: [1]

Theorem 3. *If a system of continuous functions $x_i(t) \subset E_{(a,\,b)}$ $(i = 1, 2, \cdots, n)$ forms a system T, then for any continuous function $g(t) \subset E_{(a,\,b)}$ there exists one and only one of the aggregates $\Sigma_1^n \xi_i x_i(t)$, which deviates least from $g(t)$ in sense (L).*

Proof. Indeed, if the aggregate $\Sigma_1^n a_i x_i(t)$ coincides with $g(t)$ on some interval $I_1 \subset (a, b)$ and the aggregate $\Sigma_1^n b_i x_i(t)$ coincides with $g(t)$ on the interval $I_2 \subset (a, b)$, then these intervals do not overlap; for if they did, the sum $\Sigma_1^n (b_i - a_i) x_i(t)$ would have an infinite number of zeros, and this is impossible for a system T. Therefore the set of aggregates $\Sigma_1^n \xi_i x_i(t)$ which coincide with $g(t)$ on some sub-interval of the interval (a, b) is denumerable.

On the other hand, if among the aggregates $y = \Sigma \xi_i x_i(t)$ there exist several aggregates which deviate least from $g(t)$ in sense (L) then there will exist a continuum [2] and therefore, we can find 2 different aggregates $\Sigma_1^n a_i x_i(t)$ and $\Sigma_1^n b_i x_i(t)$ from among those which do not coincide with $g(t)$ on any sub-interval of the interval (a, b). Then by virtue of proposition 1° of this paragraph

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1) Jackson established the theorem for the particular case when all $x_i(t) = t^{i-1}$ $(i = 1, 2, \cdots, n)$; however, his method together with the lemma mentioned allow us to prove the theorem for the most general case. Our proof is somewhat different from his and it seems to us simpler.

2) Cf. proposition 10° §1.

$$\operatorname{sign}\Big(g(t) - \sum_1^n a_i x_i(t)\Big) = \operatorname{sign}\Big(g(t) - \sum_1^n b_i x_i(t)\Big). \tag{34}$$

Moreover, setting

$$\alpha(t) = \lambda \operatorname{sign}\Big(g(t) - \sum_1^n a_i x_i(t)\Big), \tag{35}$$

where

$$\frac{1}{\lambda} = \min_{\xi} \int_a^b \Big| g(t) - \sum_1^n \xi_i x_i(t) \Big| \, dt,$$

we have

$$\int_a^b \alpha(t)\, x_i(t)\, dt = 0 \ (i = 1, 2, \cdots, n) \text{ and } \Big(\int_a^b \alpha(t)\, g(t)\, dt = 1\Big).^{1)} \tag{36}$$

At those points where $\alpha(t)$ has discontinuities (changes sign), $g - \Sigma a_i x_i = g - \Sigma b_i x_i = 0$ by virtue of (34) and (35) and therefore at these points $\Sigma a_i x_i = \Sigma b_i x_i$. Thus, we will reach a contradiction of the fact that the functions $x_i (i = 1, 2, \cdots, n)$ form a system T if we can show that the number of points of discontinuity of $\alpha(t)$ is greater than $n - 1$. Let us suppose that there are $m \leq n - 1$; we shall denote them by $t_1, t_2, \cdots, t_m$ and according to the previous lemma form the function

$$\Phi(t) = \sum_1^n A_i x_i(t),$$

which has the property that when t passes through any one of the points $t_1, t_2, \cdots, t_m$ and only then it changes its sign. Then

$$\sum_1^n A_i \int_a^b x_i(t)\, \alpha(t)\, dt = \int_a^b \Phi(t)\, \alpha(t)\, dt \neq 0,$$

which contradicts (36).

This proves the theorem.

In conclusion, we turn the reader's attention to the circumstance that instead

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1) The equality standing in parentheses is not used later.

of ordinary linear normed spaces we may consider linear spaces with a non-symmetric norm $\| x \|$.

In such spaces the norm $\| x \|$ satisfies the following requirements: [1]

1. $\| x \| > 0$, if $x \neq 0$,
2. $\| tx \| = t \| x \|$ for $t \geq 0$,
3. $\| x + y \| \leq \| x \| + \| y \|$,

and therefore, generally speaking, does not have the property: $\| - x \| = \| x \|$.

For example, in $E_{\mathfrak{M}}$ we obtain such a norm if we assume

$$\| x \| = \int_{\mathfrak{M}} | x(t) | \, dt + \theta \int_{\mathfrak{M}} x(t) \, dt, \tag{37}$$

where θ is a fixed number subject only to the restriction that $-1 < \theta < 1$.

The general propositions of §1 are easily carried over to linear normed spaces with non-symmetric norms, and moreover we must keep in mind that the norm of a functional $f(x)$ in such a space is still determined by the equality

$$\| f \| = \sup_{x \subset E} \frac{f(x)}{\| x \|};$$

this norm is also non-symmetric, i.e., generally speaking, $\| f \| \neq \| - f \|$.

If the norm $\| x \|$ has been defined by equality (37), which we shall denote by $\| x \|_\theta$, then obviously

$$(1 - | \theta |) \| x \|_\theta \leq \| x \|_0 \leq (1 + | \theta |) \| x \|_\theta.$$

Therefore, the norm $\| x \|_\theta$ is topologically equivalent to the norm $\| x \|_0$, i.e., the convergence of elements for one of these norms implies convergence for the other. Therefore for any $\theta (-1 < \theta < 1)$ the general form of the functional $f(x)$ in the space $E_{\mathfrak{M}}$ with the definition of the norm (37) as before will be given by the formula:

$$f(x) = \int_{\mathfrak{M}} x(t) \, \alpha(t) \, dt,$$

where $\alpha(t)$ is some measurable function with the restriction

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1) If we compare this with the definition of the norm in §1 of Article II the difference is only that in 2 we require that $t \geq 0$, where before t could be an arbitrary real number.

$$\operatorname{vrai}\max_{t\subset\mathfrak{M}} |\alpha(t)|. \tag{38}$$

However, the norm $\|f\|$ of the functional $f(x)$ does not coincide with the quantity (38) now, but with the least value L for which almost everywhere

$$-(1-\theta)L \leq \alpha(t) \leq (1+\theta)L.$$

Therefore problem A in this case must be formulated as:

A''. *Given on* $\mathfrak{M}$ *n measurable summable functions* $x_i(t)$, *find conditions on the numbers* $c_1, c_2, \cdots, c_n$, L $(\Sigma_1^n c_i^2 > 0,\ L > 0)$, *necessary and sufficient in order that there exist a measurable function* $\alpha(t)$ *for which*

$$\int_{\mathfrak{M}} x_i(t)\,\alpha(t)\,dt = c_i \quad (i = 1, 2, \cdots, n)$$

and almost everywhere

$$-L(1-\theta) \leq \alpha(t) \leq L(1+\theta) \quad (-1 < \theta < 1).$$

But this problem is nothing other than a generalization of the L_θ-problem which was treated in Chapter III of Article I.

We leave to the reader to formulate the corresponding problem B and Theorems 1, 2, 3 for the case under consideration.

We also note that in the case $\mathfrak{M} = \langle a, b\rangle$ and of continuous functions $x_i(t)$ $(i = 1, 2, \cdots, n)$ the problem A formulated is a particular case of the following moment problem:

Find necessary and sufficient conditions upon the numbers $c_1, \cdots, c_n$, *for the existence of a continuous function of bounded variation* $\sigma(t)$ *which satisfies the conditions*

$$\int_a^b x_i(t)\,d\sigma(t) = c_i \quad (i = 1, 2, \cdots, n)$$

and the condition

$$\Delta\sigma_1 \leq \Delta\sigma \leq \Delta\sigma_2 \quad (\textit{for any } \Delta t),$$

where the functions of bounded variation $\sigma_1(t)$ *and* $\sigma_2(t)$ *are given subject to the condition*

$$\Delta\sigma_1 < \Delta\sigma_2 \textit{ for any } \Delta t.$$

This problem with some additional assumptions about the functions $x_i(t)$ $(i = 1, 2, \cdots, n)$ we have previously investigated somewhat [11b]; but the methods used in this connection are beyond the scope of the present article.

CONCERNING THE THEOREM OF S. BOCHNER

N. AHIEZER

(The continuous analogue of the trigonometric moment problem)

1. We shall say that a complex function $f(t)$ defined on the entire axis $(-\infty < t < \infty)$ is positive definite in the sense of M. Mathias [32] or that it belongs to the class $\mathfrak{P}_A$ if $f(t)$ is continuous and for any natural number m, any real numbers $t_1, t_2, \cdots, t_m$, and any complex numbers $\rho_1, \rho_2, \cdots, \rho_m$ it satisfies the inequality

$$\sum_{\alpha,\,\beta=1}^{m} f(t_\alpha - t_\beta)\,\rho_\alpha \overline{\rho}_\beta \geq 0. \tag{1}$$

It follows from (1) that

$$f(-t) = \overline{f(t)}$$

and that

$$|f(t)| \leq f(0).$$

Moreover, we shall say that $f(t)$ belongs to the class $\mathfrak{P}_B$ if $f(t)$ $(-\infty < t < \infty)$ is measurable and bounded and satisfies the condition

$$f(-t) = \overline{f(t)}$$

and for any non-negative t_α, any positive η, and any real ξ:

$$\sum_{\alpha,\,\beta=1}^{m} f(t_\alpha - t_\beta)\, e^{i\xi(t_\alpha - t_\beta)} e^{-\eta(t_\alpha + t_\beta)} \geq 0.$$

Finally, we shall say the function $f(t)$ $(-\infty < t < \infty)$ belongs to the class $\mathfrak{P}_C$ if it is measurable and bounded and satisfies the condition

$$f(-t) = \overline{f(t)}$$

and for any real x and any positive y:

$$\int_0^\infty \int_0^\infty f(\alpha - \beta)\, e^{ix(\alpha-\beta)} e^{-y(\alpha+\beta)}\, d\alpha\, d\beta \geq 0.$$

It is easy to see that every function $g(t)$ which admits a representation

$$g(t) = \int_{-\infty}^{\infty} e^{iut}\, d\phi(u),$$

where $\phi(u)$ is a bounded, non-decreasing function belongs to the class $\mathfrak{P}_A$.

In fact,

$$\sum_{\alpha,\beta=0}^{m} g(t_\alpha - t_\beta)\, \rho_\alpha \overline{\rho}_\beta = \int_{-\infty}^{\infty} \left| \sum_{\alpha=1}^{m} \rho_\alpha e^{iut_\alpha} \right|^2 d\phi(u) \geq 0.$$

We shall show that this proposition admits a converse – a fact which was first proved in its entirety by S. Bochner [21a].

The classes $\mathfrak{P}_B$ and $\mathfrak{P}_C$ were first introduced in a paper of F. Riesz [36c].

2. Theorem 1. *If $f(t)$ belongs to class $\mathfrak{P}_C$, then the function*

$$\Phi(z) = \int_{0}^{\infty} e^{iuz} f(u)\, du$$

has the following two properties:

a) *$\Phi(z)$ is regular in the region $\Im(z) = y > 0$ and has a positive real part,*

b) *for any $y > 0$*

$$|\, y\Phi(iy)\,| \leq \operatorname*{vrai\,max}_{-\infty < t < \infty} |\, f(t)\,|.$$

Proof. The regularity of the function $\Phi(z)$ follows from the boundedness of the function $f(t)$.

Inequality b) is obtained from the fact that

$$|\, y\Phi(iy)\,| \leq \int_{0}^{\infty} ye^{-uy}\, |\, f(u)\,|\, du.$$

It remains to prove that for $y > 0$ the inequality $\Re\Phi(z) \geq 0$ holds.

Consider the equality

$$\frac{1}{2y} = \int_{0}^{\infty} e^{izv}\, e^{-i\overline{z}v}\, dv \quad (z = x + iy).$$

We have furthermore

$$\Phi(z) + \overline{\Phi(z)} = \int_{0}^{\infty} e^{izu} f(u)\, du + \int_{0}^{\infty} e^{-i\overline{z}u} f(-u)\, du.$$

Therefore

$$\frac{\Phi(z)+\overline{\Phi(z)}}{2y}=\int_0^\infty\int_0^\infty e^{iz(u+v)}e^{-i\bar{z}v}f(u)\,du\,dv+$$

$$+\int_0^\infty\int_0^\infty e^{-i\bar{z}(u+v)}e^{izv}f(-u)\,du\,dv=\int_0^\infty d\beta\int_\beta^\infty e^{iz\alpha}e^{-i\bar{z}\beta}f(\alpha-\beta)\,d\alpha+$$

$$+\int_0^\infty d\alpha\int_\alpha^\infty e^{iz\alpha}e^{-i\bar{z}\beta}f(\alpha-\beta)\,d\beta=\int_0^\infty\int_0^\infty f(\alpha-\beta)\,e^{ix(\alpha-\beta)}e^{-y(\alpha+\beta)}\,d\alpha\cdot d\beta\geq 0.$$

Theorem 2. *If $f(t)$ is a function of the class $\mathfrak{P}_C$, then almost everywhere the equality*

$$f(t)=\int_{-\infty}^{\infty}e^{iut}\,d\phi(u)$$

holds, where $\phi(u)$ is a non-decreasing bounded function.

Proof. On the basis of Theorem 1 and the remark to Theorem 10 of Chapter II (Article I)

$$\int_0^\infty e^{ivz}f(v)\,dv=i\int_{-\infty}^{\infty}\frac{d\phi(u)}{z+u}\quad(\Im z=y>0),$$

where $\phi(u)$ is some bounded non-decreasing function.

On the other hand,

$$\frac{i}{z+u}=\int_0^\infty e^{i(z+u)v}\,dv$$

and therefore,

$$\int_0^\infty e^{ivz}f(v)\,dv=\int_{-\infty}^{\infty}d\phi(u)\int_0^\infty e^{i(z+u)v}\,dv=\int_0^\infty e^{izv}\,dv\int_{-\infty}^{\infty}e^{iuv}\,d\phi(u).$$

This means that the two absolutely integrable functions

$$e^{-vy}f(v),\quad e^{-vy}\int_{-\infty}^{\infty}e^{iuv}\,d\phi(u)$$

have identical Fourier transforms; therefore they are almost everywhere equal, which proves our assertion.

Theorem 3. *Every function of the class $\mathfrak{P}_C$ is equivalent to some function of the class $\mathfrak{P}_A$ and every function of the class $\mathfrak{P}_A$ belongs to class $\mathfrak{P}_C$.*

Proof. If $f(t)$ belongs to class $\mathfrak{P}_C$ then $f(t)$ is equivalent to the function

$$f^*(t) = \int_{-\infty}^{\infty} e^{iut}\, d\phi(u),$$

and since

$$\sum_{\alpha,\beta=1}^{m} f^*(t_\alpha - t_\beta)\rho_\alpha \overline{\rho}_\beta = \int_{-\infty}^{\infty} \left| \sum_{\alpha=1}^{m} e^{iut_\alpha}\rho_\alpha \right|^2 d\phi(u) \geq 0,$$

the first part is proved.

Now let $f(t)$ be a function of the class $\mathfrak{P}_A$.

Then

$$\int_0^\infty \int_0^\infty e^{ix(\alpha-\beta)} e^{-\gamma(\alpha+\beta)} f(\alpha-\beta)\, d\alpha\, d\beta =$$

$$= \lim_{A\to\infty} \int_0^A \int_0^A e^{ix(\alpha-\beta)} e^{-\gamma(\alpha+\beta)} f(\alpha-\beta)\, d\alpha d\beta =$$

$$= \lim_{A\to\infty} \lim_{m\to\infty} \frac{A^2}{m^2} \sum_{\lambda,\mu=1}^{m} f\left(\frac{\lambda-\mu}{m} A\right) e^{ix\frac{\lambda-\mu}{m}A} e^{-\gamma\frac{\lambda+\mu}{m}A} \geq 0,$$

and the second part is also proved.

Comparing Theorems 2 and 3 we obtain the theorem of S. Bochner mentioned above.

We note also that by the arguments cited Theorem 1 admits a converse, namely

Theorem 4. *Every function $\Phi(z)$ which satisfies conditions* a) *and* b) *of Theorem* 1 *is represented in the form*

$$\Phi(z) = \int_0^\infty e^{iuz} f(u)\, du,$$

where $f(u)$ is some function of class $\mathfrak{P}_A$, which means of class $\mathfrak{P}_C$ also.

3. We now turn to functions of class $\mathfrak{P}_B$.

Theorem 5. *Every function of class $\mathfrak{P}_B$ is equivalent to some function of class $\mathfrak{P}_C$ and conversely.*

Proof. By Theorem 3 it is sufficient for us to prove only the first part of our assertion.

We shall trace this proof using an idea of F. Riesz [36c].

It is given that

$$\sum_{\alpha,\beta=1}^{m} f(t_\alpha - t_\beta)\, e^{i\xi(t_\alpha - t_\beta)}\, e^{-\eta(t_\alpha + t_\beta)} \geq 0$$

for any natural integer m, any non-negative $t_1, t_2, \cdots, t_m$, and $\eta > 0$, and any real ξ.

Having integrated this inequality with respect to each of the variables $t_1, t_2, \cdots, t_m$ from 0 to $A > 0$, we obtain

$$0 \leq \int_0^A \int_0^A \cdots \int_0^A \sum_{\alpha,\beta=1}^{m} f(t_\alpha - t_\beta)\, e^{i\xi(t_\alpha - t_\beta)}\, e^{-\eta(t_\alpha + t_\beta)}\, dt_1\, dt_2 \cdots dt_m =$$

$$= m(m-1)\, A^{m-2} \int_0^A \int_0^A f(u-v)\, e^{i\xi(u-v)}\, e^{-\eta(u+v)}\, du\, dv +$$

$$+ m f(0)\, A^{m-1} \int_0^A e^{-2\eta u}\, du.$$

Dividing by $m(m-1)\, A^{m-2}$ and letting m approach infinity we obtain the result that for any $A > 0$:

$$\int_0^A \int_0^A f(u-v)\, e^{i\xi(u-v)}\, e^{-\eta(u+v)}\, du\, dv \geq 0,$$

and this means that for any $\eta > 0$ and a real ξ:

$$\int_0^\infty \int_0^\infty f(u-v)\, e^{i\xi(u-v)}\, e^{-\eta(u+v)}\, du\, dv \geq 0.$$

4. Let us consider a sequence of functions

$$f_1(t),\ f_2(t), \cdots, f_n(t), \cdots$$

of class $\mathfrak{P}_A$ which converges to some function $f(t)$. It is clear that $f(t)$ belongs to the class $\mathfrak{P}_B$ and therefore almost everywhere there holds

$$f(t) = \int_{-\infty}^{\infty} e^{iut}\, d\phi(u);$$

on the other hand the equalities

$$f_n(t) = \int_{-\infty}^{\infty} e^{iut}\, d\phi_n(u); \quad (n = 1, 2, 3, \cdots)$$

hold everywhere, where $\phi(u)$, $\phi_n(u)$ are non-decreasing bounded functions.

Without loss of generality, we may take $u = 0$ as a point where the function $\phi(u)$ is continuous (otherwise we could set $\psi(u) = \phi(u - \alpha)$, $\psi_n(u) = \phi_n(u - \alpha)$, where α is some point where the function $\phi(u)$ is continuous).

Furthermore let us assume that

$$\phi(0) = 0, \quad \phi_n(0) = 0.$$

Theorem 6. (Bochner [21a]). *At every point where the function $\phi(u)$ is continuous,*

$$\lim_{n\to\infty} \phi_n(u) = \phi(u).$$

Proof. Consider the functions

$$\Phi(z) = \int_0^\infty e^{iuz} f(u)\, du = i \int_{-\infty}^{\infty} \frac{d\phi(u)}{z+u},$$

$$\Phi_n(z) = \int_0^\infty e^{iuz} f_n(u)\, du = i \int_{-\infty}^{\infty} \frac{d\phi_n(u)}{z+u} \quad (n = 1, 2, \cdots).$$

Since

$$|f_n(t)| \leq |f_n(0)| \longrightarrow |f(0)|,$$

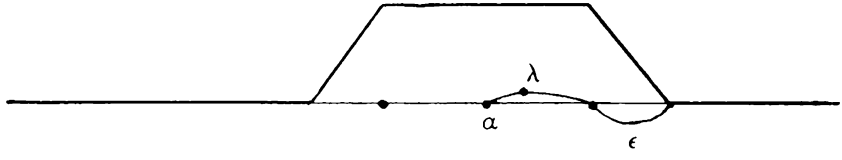
it then follows from the convergence of the sequence

$$f_1(t), \quad f_2(t), \cdots$$

by the theorem of Lebesgue that

$$\lim_{n\to\infty} \Phi_n(z) = \Phi(z) \tag{2}$$

for every z which is not real.



Let us denote by $h(t; \alpha, \lambda, \epsilon)$ the function represented in the diagram which equals zero for $t \leq \alpha - \lambda - \epsilon$ and for $t \geq \alpha + \lambda + \epsilon$, and equals one for

$\alpha - \lambda \leq t \leq \alpha + \lambda$ and which is linear in the intervals $\alpha - \lambda - \epsilon \leq t \leq \alpha - \lambda$; $\alpha + \lambda \leq t \leq \alpha + \lambda + \epsilon$ $(\epsilon > 0,\ \lambda > 0)$.

We shall show that for any $\delta > 0$ there exists an aggregate such that

$$w_m(t) = \sum_{k=1}^{m} \frac{A_k}{(t-\alpha)^2 + B_k} \quad (B_k > 0),$$

which satisfies the inequality for all $t\ (-\infty < t < \infty)$

$$| h(t;\ \alpha,\ \lambda,\ \epsilon) - w_m(t) | < \delta.$$

$h(t;\ \alpha,\ \lambda,\ \epsilon)$ is obviously a continuous function of

$$x = \frac{1}{(t-\alpha)^2 + 1}$$

in the interval $0 \leq x \leq 1$ which means by the theorem of Weierstrass that there exist constants C_k for which

$$\left| h(t;\ \alpha,\ \lambda,\ \epsilon) - \sum_{k=0}^{m} \frac{C_k}{[(t-\alpha)^2 + 1]^k} \right| < \frac{\delta}{2}$$

on the entire axis $-\infty < t < \infty$, where C_0 obviously equals zero.

The relation we have obtained may be replaced by

$$\left| h(t;\ \alpha,\ \lambda,\ \epsilon) - \sum_{k=1}^{m} \frac{C_k}{[(t-\alpha)^2 + B_1] \cdots [(t-\alpha)^2 + B_k]} \right| < \delta,$$

where all of the B_j are distinct and differ from 1 by a sufficiently small amount, but this is the required inequality.

We shall prove furthermore that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} h(u;\ \alpha,\ \lambda,\ \epsilon)\, d\phi_n(u) = \int_{-\infty}^{\infty} h(u;\ \alpha,\ \lambda,\ \epsilon)\, d\phi(u).$$

In fact

$$\left| \int_{-\infty}^{\infty} h(u;\ \alpha,\ \lambda,\ \epsilon)\, d\phi_n(u) - \int_{-\infty}^{\infty} h(u;\ \alpha,\ \lambda,\ \epsilon)\, d\phi(u) \right| \leq$$

$$\leq \delta \left\{ \int_{-\infty}^{\infty} d\phi_n(u) + \int_{-\infty}^{\infty} d\phi(u) \right\} + \left| \int_{-\infty}^{\infty} w_m(u)\, d\phi_n(u) - \int_{-\infty}^{\infty} w_m(u)\, d\phi(u) \right|.$$

The second member of the right side tends to zero as $n \to \infty$ by virtue of (2)

and the first member tends to $2\delta f(0)$, which may be made as small as desired. It is now not difficult to complete the proof. Let v be some point where the function $\phi(u)$ is continuous. We take the two functions

$$h_1(t) = h(t; \frac{v}{2}, \frac{|v|}{2} - \epsilon, \epsilon),$$

$$h_2(t) = h(t; \frac{v}{2}, \frac{|v|}{2}, \epsilon),$$

where $0 < \epsilon < \frac{|v|}{2}$.

We know that

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{\infty} h_1(t)\, d\phi_n(t) &= \int_{-\infty}^{\infty} h_1(t)\, d\phi(t), \\ \lim_{n\to\infty} \int_{-\infty}^{\infty} h_2(t)\, d\phi_n(t) &= \int_{-\infty}^{\infty} h_2(t)\, d\phi(t). \end{aligned} \tag{3}$$

On the other hand, for $v > 0$

$$\begin{aligned} &\int_{-\infty}^{\infty} h_1(t)\, d\phi_n(t) \le \phi_n(v-0) - \phi_n(+0) \le \phi_n(v) \le \\ &\le \phi_n(v+0) - \phi_n(-0) \le \int_{-\infty}^{\infty} h_2(t)\, d\phi_n(t), \\ &\int_{-\infty}^{\infty} h_2(t)\, d\phi(t) \le \phi(v+\epsilon) - \phi(-\epsilon),\ \phi(v-\epsilon) - \phi(\epsilon) \le \\ &\le \int_{-\infty}^{\infty} h_1(t)\, d\phi(t). \end{aligned} \tag{4}$$

Therefore by (3) for any $\epsilon > 0$:

$$\phi(v-\epsilon) - \phi(\epsilon) \le \underline{\lim_{n\to\infty}}\, \phi_n(v) \le \overline{\lim_{n\to\infty}}\, \phi_n(v) \le \phi(v+\epsilon) - \phi(-\epsilon),$$

and since

$$\lim_{\epsilon\to 0} \phi(\epsilon) = \lim_{\epsilon\to 0} \phi(-\epsilon) = \phi(0) = 0,$$

$$\lim_{\epsilon\to 0} \phi(v-\epsilon) = \lim_{\epsilon\to 0} \phi(v+\epsilon) = \phi(v),$$

then

$$\phi(v) = \lim_{n \to \infty} \phi_n(v).$$

For $v < 0$ the proof is analogous, the only difference is that inequality (4) is replaced by

$$\int_{-\infty}^{\infty} h_1(t)\, d\phi_n(t) \leq \phi_n(-0) - \phi_n(v+0) \leq -\phi_n(v) \leq$$

$$\leq \phi_n(+0) - \phi_n(v-0) \leq \int_{-\infty}^{\infty} h_2(t)\, d\phi_n(t),$$

$$\int_{-\infty}^{\infty} h_2(t)\, d\phi(t) \leq \phi(+\epsilon) - \phi(v-\epsilon), \quad \phi(-\epsilon) - \phi(v+\epsilon) \leq \int_{-\infty}^{\infty} h_1(t)\, d\phi(t).$$

CONCERNING A SPECIAL CLASS OF ENTIRE AND MEROMORPHIC FUNCTIONS

M. KREĬN

The present article is devoted to the study begun by Stieltjes [39a] and continued by Grommer [27] of a class of entire and meromorphic functions by the methods of the theory of moments.

Other authors after Stieltjes and Grommer have studied this class of functions by various methods; in particular, N. G. Čebotarev [17] obtained interesting additions to Grommer's research by algebraic considerations.

M. Fujiwara [26b] attempted to apply some algebraic methods which Hermite originated to the theory of this class of functions, but he did not obtain sufficiently general results and, furthermore, he erred in his deductions. [1]

It seems to us that the most natural and direct methods for this complex of questions are nevertheless the transcendental methods of the moment problem and the theory of bounded functions of a complex variable in general. In every case, as we shall show, one manages by this method to bring the solution of a whole series of questions to its logical conclusion.

§1

1. Let the sequence s_k $(k = 0, 1, \cdots, 2n-1)$ be positive definite [2] in the interval $(-\infty, \infty)$, which implies that the quadratic form

$$\sum_{i,k=0}^{n-1} s_{i+k}\xi_i\bar{\xi}_k = \int_{-\infty}^{\infty}\left|\sum_0^{n-1}\xi_i t^i\right|^2 d\sigma$$

is positive. [3] Let us consider the orthonormal polynomials

$$\hat{P}_k(t) = \frac{1}{\sqrt{D_k D_{k-1}}}\begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ \cdots & \cdots & \cdots & \cdots \\ s_{k-1} & s_{k-2} & \cdots & s_{2k-1} \\ 1 & t & \cdots & t^k \end{vmatrix} \quad \left(D_k = |s_{i+j}|_0^k,\ k = 0, 1, \ldots, n-1,\ D_{-1} = 1\right),$$

1) Cf. §6. In this paragraph we shall also correct one error by Grommer.

2) Concerning positive definite sequences see Chapter I of Article I. Here, in particular, we shall use Theorems 1a – 2a, Chapter II of Article I; their proof may be also found in Article II, pages 133 and 139.

3) For conciseness we shall say *positive* instead of *positive definite*.

i.e., the polynomials of progressively increasing degrees which are defined by the relations

$$\mathfrak{S}\{\hat{P}_i \hat{P}_k\} = \begin{cases} 1 (i = k) \\ 0 (i \neq k) \end{cases} \quad (i, k = 0, 1, \cdots, n-1). \tag{1}$$

Let

$$t\hat{P}_i = \sum_{k=0}^{n-1} a_{ik} \hat{P}_k \quad (i = 0, 1, \cdots, n-2); \tag{2}$$

multiplying this equality by $\hat{P}_j$ and applying the functional $\mathfrak{S}$ to both sides, by virtue of relations (1) we obtain

$$a_{ik} = \mathfrak{S}\{t\hat{P}_i \hat{P}_k\}.$$

Here k takes on the values $0, 1, \cdots, n-1$, and i the values $0, 1, \cdots, n-2$; moreover, we define the quantities a_{ik} of this formula for all the values $i, k = 0, 1, \cdots, n-1$.

It is obvious that $a_{ik} = a_{ki} (i, k = 0, 1, \cdots, n-1)$. On the other hand, it follows from (2) that $a_{ik} = 0$ for $k > i + 1$; from this in general

$$a_{ik} = 0 \text{ for } |i - k| > 1.$$

Thus, the matrix $\| a_{ik} \|_0^{n-1}$ is a symmetric Jacobi matrix.

It is easy also to verify that

$$a_k = a_{kk} = \frac{D_k'}{D_k} - \frac{D_{k-1}'}{D_{k-1}} \quad (k = 0, 1, \ldots, n-1)$$

$$b_k = a_{kk+1} = \frac{\sqrt{D_{k-1} D_{k+1}}}{D_k} \quad (k = 0, 1, \ldots, n-2),$$

where

$$D_{-1}' = 0, \; D_0' = s_1, \; D_k' = \begin{vmatrix} s_0 & s_1 & \cdots & s_{k-1} & s_{k+1} \\ s_1 & s_2 & \cdots & s_k & s_{k+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_k & s_{k+1} & \cdots & s_{2k-1} & s_{2k+1} \end{vmatrix} \quad (k = 0, 1, 2, \ldots).$$

For this it is necessary to equate the coefficients of the two highest terms in the relation

$$t\hat{P}_i(t) = b_{i-1} \hat{P}_{i-1} + a_i \hat{P}_i + b_i \hat{P}_{i+1} \quad (i = 0, 1, \cdots, n-2), \tag{3}$$

bearing in mind that

$$\hat{P}_i(t) = \sqrt{\frac{D_{i-1}}{D_i}}\, t^i - \frac{D'_{i-1}}{\sqrt{D_i D_{i-1}}}\, t^{i-1} + \cdots \; (i = 0, 1, \cdots, n-2).$$

We shall now show that conversely every symmetric Jacobi matrix $\| a_{ik} \|_0^{n-1}$, for which

$$a_i = a_{ii} \gtrless 0 \; (i = 0, 1, \cdots, n-1), \quad b_i = a_{ii+1} > 0 \; (i = 0, 1, \cdots, n-2)$$

is generated by some positive definite sequence $s_k \; (k = 0, 1, \cdots, 2n-1)$ and this sequence is uniquely defined if we select arbitrarily a positive s_0.

For this we shall set

$$\hat{P}_0 = \frac{1}{\sqrt{s_0}},$$

where s_0 is an arbitrarily selected positive number and define from the recurrence formulas (3) the following polynomials $\hat{P}_i(t) \; (i = 1, 2, \cdots, n-1)$.

Now let

$$G(t) = \sum_0^{n-1} \xi_i \hat{P}_i(t)$$

$$H(t) = \sum_0^{n-1} \eta_i \hat{P}_i(t)$$

be two arbitrary polynomials of degree $\leq n-1$. We shall define the "scalar product" (G, H) of these polynomials by the equality

$$(G, H) = \sum_0^{n-1} \xi_i \eta_i.$$

It is obvious that

1. $(G, H) = (H, G)$
2. $(G_1 + G_2, H) = (G_1, H) + (G_2, H)$
3. $(\alpha G, H) = \alpha (G, H)$
4. $(G, G) > 0$, if $G(t) \not\equiv 0$.

Since the matrix $\| a_{ik} \|_0^{n-1}$ is symmetric, it follows from (3) that

$$(t\hat{P}_i, \hat{P}_k) = (\hat{P}_i, t\hat{P}_k) = a_{ik} (i, k = 0, 1, \cdots, n-2),$$

and therefore,

$$(tG, H) = (G, tH)$$

and in general

$$(FG, H) = (G, FH),$$

if the degrees of the products FG and FH are not higher than $n - 1$.

We shall now set

$$s_{g+h} = (t^g, t^h) \quad (g, h = 0, 1, \cdots, n-1). \tag{4}$$

This defines a certain sequence $s_0, s_1, \cdots, s_{2n-2}$. We now introduce the functional $\mathfrak{S}$ which corresponds to it. Since

$$(t^g, t^h) = \mathfrak{S}\{t^{g+h}\}, \tag{5}$$

in general for any two polynomials G and H with degrees not greater than $n - 1$ there holds the equality

$$\mathfrak{S}\{GH\} = (G, H);$$

in particular,

$$\mathfrak{S}\{\hat{P}_i \hat{P}_k\} = (\hat{P}_i, \hat{P}_k) = \begin{cases} 1 \ (i = k) \\ 0 \ (i \neq k). \end{cases} \tag{6}$$

Since, moreover, by property 4° of the product (G, H), the form

$$\sum_0^{n-1} s_{i+k}\xi_i\xi_k = \mathfrak{S}\left\{\left[\sum_0^{n-1}\xi_i t^i\right]^2\right\} = \left[\sum_0^{n-1}\xi_i t^i, \sum_0^{n-1}\xi_i t^i\right]$$

is positive, then the sequence $s_0, s_1, \cdots, s_{2n-2}$ is positive definite.

We shall now define s_{2n-1} so that

$$\mathfrak{S}\{t\hat{P}_{n-1}^2(t)\} = a_{n-1\ n-1} = a_{n-1}. \tag{7}$$

Then by (3) and (6) we shall have

$$\mathfrak{S}\{t\hat{P}_i, \hat{P}_k\} = a_{ik} \text{ for all } i, k = 0, 1, \cdots, n-1.$$

Thus, the matrix $\| a_{ik} \|_0^{n-1}$ is in fact generated by some positive definite sequence $s_0, s_1, \cdots, s_{2n-1}$. This sequence is uniquely determined for given $s_0 > 0$, because if it generates the matrix $\| a_{ik} \|_0^{n-1}$, then relation (7) is valid and so is (5) and consequently, (4).

This proves our assertion. [1)]

We note now that if one defines the non-singular transformation

$$\eta_i = \sum_{k=0}^{n-1} a_{ik}\xi_k \quad (i = 0, 1, \cdots, n-1) \tag{8}$$

by the equality

$$\sum_{0}^{n-1} \xi_i t^i = \sum_{0}^{n-1} \eta_i \hat{P}_i(t),$$

then

$$\sum_{i,k=0}^{n-1} s_{i+k+1}\xi_i\xi_k = \sum_{i,k=0}^{n} a_{ik}\eta_i\eta_k .$$

In fact,

$$\sum_{i,k=0}^{n-1} s_{i+k+1}\xi_i\xi_k = \mathfrak{S}\left\{ t\left[\sum_{0}^{n-1}\xi_i t^i\right]^2\right\} = \mathfrak{S}\left\{ t\left[\sum_{0}^{n-1}\eta_i \hat{P}_i(t)\right]^2\right\} =$$

$$= \sum_{i,k=0}^{n-1} a_{ik}\eta_i\eta_k .$$

§2

As we already know, every infinite sequence of real numbers

$$s_0 > 0,\ a_0,\ b_0,\ a_1,\ b_1, \cdots \quad (b_i > 0,\ i = 0, 1, 2, \cdots) \tag{9}$$

generates some sequence positive definite in the interval $(-\infty, \infty)$. We shall show that:

There exists a finite interval in which the sequence of numbers $s_0, s_1, \cdots$ *is positive definite, if and only if the following two conditions are fulfilled:* 1° *the sequence is positive definite in the interval* $(-\infty, \infty)$, *i.e., all* $D_k > 0$ $(k = 0, 1, 2, \cdots)$,

---

1) The reader who has been attentively following our arguments probably has noticed that we have proved somewhat more, to wit: for a given $s_0 > 0$, the first m quantities in the sequence

$$a_0, b_0, a_1, b_1, \cdots$$

determine the first m quantities in the sequence

$$s_1, s_2, \cdots$$

and conversely.

and 2° *the corresponding sequence* $a_0, b_0, a_1, b_1, \cdots$ *is bounded.*

Indeed, the necessity of the first condition is obvious. With this condition satisfied we may construct an orthonormal sequence $\hat{P}_0(t), \hat{P}_1(t), \cdots$. Since

$$\sum_0^{n-1} (\lambda \epsilon_{ik} \pm a_{ik}) \eta_i \eta_k = \mathfrak{S}\left\{ (\lambda \pm t) \left[\sum_0^{n-1} \eta_i \hat{P}_i(t) \right]^2 \right\} \tag{I}$$

$$\sum_0^{n-1} (\lambda s_{i+k} \pm s_{i+k+1}) \xi_i \xi_k = \mathfrak{S}\left\{ (\lambda \pm t) \left[\sum_0^{n-1} \xi_i t^i \right]^2 \right\}, \tag{II}$$

the form (I) is changed into form (II) by the non-singular transformation (8). According to a theorem proved earlier, [1] the sequence $s_0, s_1, \cdots$ is positive definite in the interval $(-R, R)$ if and only if for any n, the forms

$$\sum_0^{n-1} (R s_{i+k} \pm s_{i+k+1}) \xi_i \xi_k$$

and consequently the forms

$$\sum_0^{n-1} (R \epsilon_{ik} \pm a_{ik}) \eta_i \eta_k$$

are positive. But if the latter forms are positive, then

$$R \pm a_{ii} = R \pm a_i > 0 \quad (i = 0, 1, \cdots)$$

and

$$\begin{vmatrix} R \pm a_{ii} & \pm a_{ii+1} \\ \pm a_{i+1i} & R \pm a_{i+1i+1} \end{vmatrix} = \begin{vmatrix} R \pm a_i & \pm b_i \\ \pm b_i & R \pm a_{i+1} \end{vmatrix} \geqslant 0 \quad (i = 0, 1, 2, \ldots).$$

From these inequalities it follows that

$$|a_i| < R, \ |b_i| < 2R \quad (i = 0, 1, 2, \cdots).$$

Conversely, now let the sequence (9) be bounded. We take $R > \sup|a_i| + 2\sup|b_i|$; for such an R

$$\sum_0^{n-1} (R \epsilon_{ik} \pm a_{ik}) \eta_i \eta_k = \sum_0^{n-1} (R \pm a_i) \eta_i^2 \pm 2 \sum_0^{n-2} b_i \eta_i \eta_{i+1} \geq$$
$$\geq \sum_0^{n+1} (R \pm a_i - 2|b_i|) \eta_i^2 > 0,$$

---

1) Article I, pages 34 – 36.

because

$$2b_i\eta_i\eta_{i+1} < |b_i|(\eta_i^2 + \eta_{i+1}^2) \quad (i = 0, 1, \cdots).$$

This proves our assertion. By results obtained earlier it is equivalent to the following well-known theorem.

Theorem 1 (Toeplitz [41b]). *In order for a sequence of real numbers* $s_0, s_1, s_2, \cdots$ *to admit a representation*

$$s_k = \int_{-\infty}^{\infty} t^k \, d\sigma(t) \quad (k = 0, 1, 2, \cdots), \tag{10}$$

where the non-decreasing function $\sigma(t)$ $(-\infty < t < \infty)$ *satisfies the two conditions:* 1° *the number of growth points of the function is infinite,* 2° *all growth points of this function are concentrated in a finite segment, it is necessary and sufficient that all of the determinants*

$$D_k = \begin{vmatrix} s_0 \cdots s_k \\ \cdots \\ s_k \cdots s_{2k} \end{vmatrix} \qquad (k = 0, 1, 2, \ldots)$$

are positive and the set of numbers

$$a_k = \frac{D_k'}{D_k} - \frac{D_{k-1}'}{D_{k-1}}, \quad b_k = \frac{\sqrt{D_{k-1}D_{k+1}}}{D_k} \quad (k = 0, 1, 2, \ldots)$$

is bounded.

In the following paragraphs we shall find criteria for all cluster points of the set of growth points of the function $\sigma(t)$ in representation (10) to be contained in a finite set specified in advance.

§3

1. Let $s_0, s_1, s_2, \cdots$ be an infinite sequence of real numbers which satisfy the conditions of Theorem 1. Therefore, there can be found an $R > 0$ and a non-decreasing function $\sigma(t)$ $(-R \leq t \leq R)$, having an infinite set of growth points such that:

$$s_k = \int_{-R}^{R} t^k \, d\sigma(t) \quad (k = 0, 1, \cdots).$$

Let us denote by C_R the collection of all real continuous functions in the interval $< -R, R >$. We shall define for every pair of functions $\phi \subset C_R$, $\psi \subset C_R$ a scalar product (ϕ, ψ) by the formula

$$(\phi, \psi) = \int_{-R}^{R} \phi(t)\psi(t)\, d\sigma(t).$$

Moreover, we let

$$|\phi|_\sigma = \sqrt{(\phi, \phi)}.$$

It is obvious that $|\phi|_\sigma \geq 0$ and the equality sign will hold if and only if $\phi(t) \equiv 0$ on the set of points of growth of the function $\sigma(t)$. Let us denote this set by E_σ. We shall agree to write

$$\phi \overset{\sigma}{=} \psi, \text{ if } |\phi - \psi|_\sigma = 0 \quad (\phi, \psi \subset C_R).$$

The scalar product (ϕ, ψ) has the usual properties:

1. $(\phi, \psi) = (\psi, \phi)$,
2. $(\lambda\phi, \psi) = \lambda(\phi, \psi)$
3. $(\phi_1 + \phi_2, \psi) = (\phi_1, \psi) + (\phi_2, \psi)$
4. $(\phi, \phi) > 0$, if $\phi \overset{\sigma}{\neq} 0$.

From this it is easy to deduce Schwarz's inequality

$$|(\phi, \psi)| \leq |\phi|_\sigma |\psi|_\sigma,$$

in which the equality holds if and only if either $|\phi|_\sigma |\psi|_\sigma = 0$, or $\psi \overset{\sigma}{=} C\phi$ (C is a constant).

We recall also that from Schwarz's inequality there follows the inequality

$$|\phi + \psi|_\sigma \leq |\phi|_\sigma + |\psi|_\sigma;$$

indeed,

$$|\phi + \psi|_\sigma^2 = (\phi, \phi) + 2(\phi, \psi) + (\psi, \psi) \leq |\phi|_\sigma^2 + 2|\phi|_\sigma |\psi|_\sigma + |\psi|_\sigma^2.$$

We now introduce several definitions.

A sequence $\{\phi_n\}$ of elements from C_R will be called σ-*convergent* if

$$|\phi_m - \phi_n|_\sigma \to 0 \text{ for } m, n \to \infty.$$

Also, we shall say that the sequence $\{\phi_n\}$ has a σ-limit $\phi \subset C_R$ and write

$$\phi_n \overset{\sigma}{\to} \phi \text{ or } \phi \overset{\sigma}{=} \lim \phi_n,$$

if

$$|\phi - \phi_n|_\sigma \to 0.$$

It is easy to see that if a sequence $\{\phi_n\}$ has a σ-limit it is σ-convergent, but the converse is not in general true.

It should also be clear to the reader what we mean by the equality

$$\phi \overset{\sigma}{=} \phi_1 + \phi_2 + \cdots, \tag{11}$$

where ϕ and $\phi_i (i = 1, 2, \cdots)$ are elements from C_R.

For any $\epsilon > 0$, for every function $\phi \subset C_R$ we can determine by the theorem of Weierstrass a number n and constants $\xi_0^0, \xi_1^0, \cdots, \xi_n^0$ such that

$$|\phi(t) - \sum_0^n \xi_k^0 \hat{P}_k(t)| < \epsilon \quad (-R \le t \le R).$$

But then

$$\left|\varphi - \sum_0^n \xi_k^0 \hat{P}_k\right|_\sigma = \sqrt{\int_{-R}^{R} \left\{\varphi(t) - \sum_0^n \xi_k^0 \hat{P}_k(t)\right\}^2 d\sigma(t)} < \\ < \epsilon\sqrt{\sigma(R) - \sigma(-R)}. \tag{12}$$

On the other hand

$$\min_{\xi} \left|\phi - \sum_0^n \xi_k \hat{P}_k\right|_\sigma^2 = |\phi|_\sigma^2 - \sum_0^n c_k^2, \tag{13}$$

where

$$c_k = (\phi, \hat{P}_k) = \int_{-R}^{R} \phi(t) \hat{P}_k(t)\, d\sigma(t) \quad (k = 0, 1, \cdots, n);$$

and where the minimum in (13) is attained for $\xi_k = c_k (k = 0, 1, \cdots)$. Combining (13) with (12) we reach the conclusion that for every function $\phi \subset C_R$ the equality

$$\phi \overset{\sigma}{=} \sum_0^\infty c_k \hat{P}_k$$

is valid, where

$$c_k = (\phi, \hat{P}_k) \quad (k = 0, 1, \cdots).$$

We note that if $\phi_n \overset{\sigma}{\to} \phi$ and $\psi_n \overset{\sigma}{\to} \psi$, then $(\phi_n, \psi_n) \to (\phi, \psi)$. In fact,

$$(\phi, \psi) - (\phi_n, \psi_n) = (\phi - \phi_n, \psi) + (\phi_n, \psi - \psi_n),$$

but

$$|(\phi - \phi_n, \psi)| \le |\phi - \phi_n|_\sigma |\psi|_\sigma \text{ and } |(\phi_n, \psi - \psi_n)| \le |\phi_n|_\sigma |\psi - \psi_n|_\sigma.$$

It follows from this that equality (11) may be scalar multiplied term by term

by any element $\psi \subset C_R$, i.e., (11) implies

$$(\phi, \psi) = \sum_1^\infty (\phi_k, \psi).$$

In particular, if the decomposition

$$\phi \overset{\sigma}{=} \sum_0^\infty \xi_k \hat{P}_k \tag{14}$$

is scalar multiplied by $\hat{P}_j$ then by using the orthogonality relations we obtain the equalities $\xi_j = c_j$ $(j = 0, 1, \cdots)$, which proves the unicity of decomposition (14).

We note also that from the equalities

$$\phi \overset{\sigma}{=} \sum_0^\infty c_k \hat{P}_k, \quad \psi \overset{\sigma}{=} \sum_0^\infty d_k \hat{P}_k$$

it follows that

$$(\phi, \psi) = \sum_0^\infty c_k d_k,$$

because

$$(\sum_0^n c_k \hat{P}_k, \sum_0^n d_k \hat{P}_k) = \sum_0^n c_k d_k.$$

2. An operator U which associates some element $\psi = U\phi \subset C_R$ to each element $\phi \subset C_R$ is called linear if

1. $U(\phi_1 + \phi_2) = U\phi_1 + U\phi_2$.
2. $U(\lambda\phi) = \lambda U\phi$.

Let

$$U\hat{P}_j \overset{\sigma}{=} \sum_0^\infty u_{ik} \hat{P}_k \quad (i = 0, 1, \cdots);$$

then by $\| U \|$ we shall mean the infinite matrix $\| u_{ik} \|_0^\infty$. The elements u_{ik} $(i, k = 0, 1, \cdots)$ may be determined as follows:

$$u_{ik} = (U\hat{P}_i, \hat{P}_k) \quad (i, k = 0, 1, 2, \cdots).$$

We define the operator A by the equality

$$A\phi = t\phi(t). \tag{15}$$

Since

$$t\hat{P}_i(t) = b_{i-1} \hat{P}_{i-1}(t) + a_i \hat{P}_i(t) + b_i \hat{P}_{i+1}(t) \quad (i = 0, 1, 2, \cdots), \tag{16}$$

then

$$\| A \| = \| a_{ik} \|_0^\infty = \begin{vmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 & 0 \\ b_0 & a_1 & b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot \\ 0 & 0 & & & & \cdot & \cdot \end{vmatrix}.$$

It is clear what shall be meant by the sum, difference or product of two operators and what degree and polynomial of an operator mean.

Henceforth, we must deal with the operator

$$g(A), \text{ where } g(x) = x^p + \alpha_1 x^{p-1} + \cdots + \alpha_p.$$

It is obvious that by definition (15)

$$g(A)\phi = g(t)\phi(t) \text{ for any } \phi \subset C_R.$$

Let

$$g(A)\hat{P}_i = g(t)\hat{P}_i(t) = \sum_{k=0}^{\infty} g_{ik}\hat{P}_k \quad (i = 0, 1, 2, \cdots),$$

i.e.,

$$\| g(A) \| = \| g_{ik} \|_0^\infty.$$

From (16) it is not difficult to obtain

$$g_{ik} = 0 \text{ for } | i - k | > p;$$

and furthermore it is not difficult to see that

$$\| g(A) \| = g(\| A \|) = \| A \|^p + \alpha_1 \| A \|^{p-1} + \cdots + \alpha_p I,$$

where the power $\| A \|^k$ $(k = 1, 2, \cdots)$ is computed by the usual rules of multiplication of matrices, and $I = \| \delta_{ik} \|_0^\infty$ is the identity matrix.

We shall introduce one more definition. A linear operator U is called *completely continuous* if it transforms each σ-bounded sequence $\{\phi_n\}$ $(| \phi_n |_\sigma < M, n = 1, 2, \cdots)$ into a sequence $\{U\phi_n\}$, from which we may select a σ-convergent subsequence.

We shall show that:

1°. The operator $g(A)$, where $g(x)$ is a polynomial, is completely continuous

if and only if

$$\lim_{i,\, k\to\infty} g_{ik} = 0. \tag{17}$$

Indeed, if condition (17) is not satisfied then $\mu > 0$ and a sequence of pairs of integers

$$i_1,\ k_1 < i_2,\ k_2 < i_3,\ k_3 < \cdots$$

may be found such that

$$|g_{i_s k_s}| > \mu \quad (s = 1, 2, \cdots). \tag{18}$$

The pairs i_s, k_s $(s = 1, 2, \cdots)$ may, moreover, be chosen so that they fulfill the condition

$$|i_s - i_{s+1}| > 2p \quad (s = 1, 2, \cdots). \tag{19}$$

Let us consider the sequence of polynomials $\hat{P}_{i_s}(t)$ $(s = 1, 2, \cdots)$. This is a σ-bounded sequence, since

$$|\hat{P}_{i_s}(t)|_\sigma = 1 \quad (s = 1, 2, \cdots).$$

On the other hand, by (18) and (19) for any integers s and t

$$|g(A)\hat{P}_{i_s} - g(A)\hat{P}_{i_t}|_\sigma^2 = \Big|\sum_{k=i_s-p}^{i_s+p} g_{i_s k}\hat{P}_k - \sum_{k=i_t-p}^{i_t+p} g_{i_t k}\hat{P}_k\Big|_\sigma^2 =$$

$$= \sum_{k=i_s-p}^{i_s+p} g_{i_s k}^2 + \sum_{k=i_t-p}^{i_t+p} g_{i_t k}^2 > 2\mu^2.$$

Therefore, it is impossible to select a σ-convergent subsequence from the sequence $g(A)\hat{P}_{i_s}$ and the operator $g(A)$ is not completely continuous.

Now, on the contrary, let condition (17) be satisfied. We shall show that the operator $g(A)$ is completely continuous in this case.

For this we consider an arbitrary σ-bounded sequence $\{\phi_n\}$ of elements of C_R $(|\phi_n|_\sigma < M,\ n = 1, 2, \cdots)$. Let

$$\phi_n \overset{\sigma}{=} \sum_{k=0}^{\infty} c_k^{(n)}\hat{P}_k,$$

and therefore,

$$|\phi_n|_\sigma^2 = \sum_{k=0}^{\infty} (c_k^{(n)})^2 < M^2 \quad (n = 1, 2, \cdots). \tag{20}$$

By virtue of these inequalities numbers $n_1 < n_2 < \cdots$ can be determined by Cantor's diagonal process so that there exist the limits

$$\lim_{\nu \to \infty} c_k^{(n_\nu)} = c_k \quad (k = 0, 1, 2, \cdots).$$

We note, on the other hand, that if

$$\phi \overset{\sigma}{=} \sum_{i=0}^{\infty} \xi_i \hat{P}_i,$$

then in the expansion

$$\psi = g(A)\phi = \sum_{i=0}^{\infty} \eta_i \hat{P}_i$$

the quantities η_i $(i = 0, 1, \cdots)$ take the following form:

$$\eta_i = (g(A)\phi, \hat{P}_i) = (g(t)\phi, \hat{P}_i) = (\phi, g(t)\hat{P}_i) = (\phi, \sum_{k=i-p}^{i+p} g_{ik}\hat{P}_k) =$$

$$= \sum_{k=i-p}^{i+p} g_{ik}\xi_k.$$

Therefore,

$$\eta_i^2 \leq \sum_{k=i-p}^{i+p} g_{ik}^2 \sum_{k=i-p}^{i+p} \xi_k^2$$

which means if

$$|g_{ik}| < \epsilon \text{ for } i > N_\epsilon, \tag{21}$$

then

$$\sum_{i=N+1}^{\infty} \eta_i^2 \leq (2p+1)^2 \epsilon^2 \sum_{k=0}^{\infty} \xi_k^2 = (2p+1)^2 \epsilon^2 |\phi|_\sigma^2. \tag{22}$$

Thus, if we set

$$\psi_\nu = g(A)\phi_{n_\nu} \overset{\sigma}{=} \sum_{i=0}^{\infty} \eta_i^{(\nu)} \hat{P}_i, \tag{23}$$

then

$$\eta_i^{(\nu)} = \sum_{k=i-p}^{i+p} g_{ik} c_k^{(n_\nu)} \quad (i = 0, 1, 2, \cdots),$$

and hence

$$\lim_{\nu\to\infty}\eta_i^{(\nu)} = \sum_{k=i-p}^{i+p} g_{ik}\,c_k \quad (i = 0,\ 1,\ 2,\cdots). \tag{24}$$

Besides, if for a given $\epsilon > 0$, N_ϵ is chosen so that condition (21) is satisfied, then by (20) and (22)

$$\sum_{i=N+1}^{\infty} (\eta_i^{(\nu)})^2 < (2p+1)^2\,\epsilon^2\,M^2. \tag{25}$$

Relations (24) and (25) allow us to assert that the sequence $\{\psi_n\}$ is σ-convergent, and this means the operator $g(A)$ is completely continuous.

In fact, by (23)

$$|\psi_\mu - \psi_\nu|_\sigma^2 = \sum_{i=0}^{\infty} |\eta_i^{(\mu)} - \eta_i^{(\nu)}|^2 \le \sum_{i=0}^{N} |\eta_i^{(\mu)} - \eta_i^{(\nu)}|^2 + 2\sum_{i=N+1}^{\infty} |\eta_i^{(\mu)}|^2 +$$

$$+ 2\sum_{i=N+1}^{\infty} |\eta_i^{(\nu)}|^2 \le \sum_{i=0}^{N} |\eta_i^{(\mu)} - \eta_i^{(\nu)}|^2 + 4(2p+1)^2\,\epsilon^2\,M^2.$$

And therefore, if L is chosen so that

$$\sum_{i=0}^{N} |\eta_i^{(\mu)} - \eta_i^{(\nu)}|^2 < \epsilon^2 \text{ for } \mu,\ \nu > L,$$

then according to (24) it is always possible that

$$|\psi_\mu - \psi_\nu|^2 < \epsilon^2[1 + 4(2p+1)^2\,M^2] \text{ for } \mu,\ \nu > L.$$

Assertion 1° is proved.

3. In subsequent reasoning we shall assume that $g(t)$ is, in general, some function which is continuous in the interval $<-R,\ R>$ and as a particular case it may be a polynomial. With this assumption we shall by convention mean by $g(A)$ the linear operator defined by the equality

$$g(A)\,\phi = g(t)\,\phi(t) \quad (\phi \subset C_R).$$

We shall agree, moreover, if E is some set from the interval $<-R,\ R>$, to mean by $g(\mathrm{E})$ the collection of all values of the function $g(x)$ on E, counting each value as many times as it is assumed.[1] Besides criterion 1° that an operator $g(A)$ be completely continuous there is another such:

2°. *A linear operator $g(A)$, where $g(x)$ is some function from C_R, is*

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1) The last condition plays a role in the definition of cluster point of the set $g(\mathrm{E}_\sigma)$.

completely continuous if and only if the set $g(E_\sigma)$ *has as its only cluster point the point zero.*

Indeed, let us assume the set $g(E_\sigma)$ has at least one cluster point different from zero. Then there exists a $\mu > 0$ such that for an infinite set of points from E_σ the inequality $|g(x)| > \mu$ is satisfied.

Since the function $g(x)$ is continuous, a subinterval I of the interval $<-R, R>$ can be found in which there is an infinite number of points of E_σ and in which there is satisfied the inequality $|g(x)| > \mu\ (x \subset I)$.

We shall denote by C_I the set of those functions $\phi(x) \subset C_R$, which vanish everywhere outside of the interval I. We note that

$$|g(A)\phi|_\sigma > \mu |\phi|_\sigma, \text{ if } \phi \subset C_I. \tag{26}$$

In fact, if $\phi \subset C_I$, then

$$|g(A)\phi|_\sigma^2 = \int_{-R}^{R} g^2(x)\phi^2(x)\,d\sigma(x) = \int_I g^2(x)\phi^2(x)\,d\sigma(x) >$$

$$> \mu^2 \int_I \phi^2(x)\,d\sigma(x) = \mu^2 |\phi|_\sigma^2.$$

On the other hand, since inside I the function $\sigma(x)$ has an infinite number of points of growth, C_I contains an infinite set of σ-linearly independent functions and, therefore, contains some infinite orthonormal system of functions. [1] For this system

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1) Indeed, let the functions $\psi_1, \psi_2, \cdots, \psi_n$ $(\{\psi_n\} \subset C_I)$ be σ-linearly independent functions. Then the form

$$\sum_{i,k=1}^{n} (\psi_i, \psi_k)\xi_i\xi_k = \int_I \left[\sum_1^n \xi_i\psi_i\right]^2 d\sigma(t) \quad (n = 1, 2, \cdots)$$

is positive and hence the determinant of the form $\Delta_n = |(\psi_i, \psi_k)|_1^n > 0$ $(n = 1, 2, \cdots)$. Then let

$$\varphi_n = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} (\psi_1, \psi_1) \cdots (\psi_1, \psi_{n-1}) & \psi_1 \\ (\psi_2, \psi_1) \cdots (\psi_2, \psi_{n-1}) & \psi_2 \\ \cdots\cdots\cdots\cdots & \cdots \\ (\psi_n, \psi_1) \cdots (\psi_n, \psi_{n-1}) & \psi_n \end{vmatrix} \quad (n = 1, 2, \ldots;\ \Delta_0 = 1).$$

It is easy to show that the sequence $\{\phi_n\}_1^\infty$, is an orthonormal system of functions and it is obvious that $\{\phi_n\}_1^\infty \subset C_I$.

$$(\phi_i, \phi_k) = \begin{cases} 1\ (i = k) \\ 0\ (i \neq k) \end{cases} \quad (i, k = 1, 2, \cdots),$$

and therefore,

$$|\phi_m|_\sigma = 1\ (m = 1, 2, \cdots),\ |\phi_m - \phi_n|_\sigma = \sqrt{2}\ (m \neq n;\ m, n = 1, 2, \cdots).$$

But then by virtue of (26)

$$|g(A)\phi_m - g(A)\phi_n|_\sigma = |g(A)(\phi_m - \phi_n)| > \mu |\phi_m - \phi_n| = \mu\sqrt{2}\ (m \neq n),$$

and it is apparent that the operator $g(A)$ is not completely continuous.

Now let to the contrary, the set $g(E_\sigma)$ have a single cluster point equal to 0. Then, if at some point $t_0 \subset E_\sigma$ the function $g(t)$ has the value $s_0 = g(t_0)$ then the point t_0 is a jump point of the function $\sigma(t)$, i.e., $\sigma(t_0 + 0) - \sigma(t_0 - 0) > 0$, since otherwise the point t_0 would be a cluster point of the set E_σ which would mean that the point $s_0 \neq 0$ was a cluster point of the set $g(E_\sigma)$ which contradicts our assumptions.

Renumbering those points of E_σ, where $g(t) \neq 0$; we thus obtain a sequence $t_1, t_2, \cdots$, and by our assumptions regarding $g(E_\sigma)$

$$\lim_{k \to \infty} g(t_k) = 0.$$

For brevity we set

$$\rho_k = \sigma(t_k + 0) - \sigma(t_k - 0) \quad (k = 1, 2, \cdots).$$

If a point t does not belong to the sequence $\{t_k\}$ then either it does not belong to E_σ, or $g(t) = 0$ at this point. Therefore one has the equality

$$|g(A)\phi|_\sigma^2 = \int_{-R}^{R} g^2(t)\phi^2(t)\, d\sigma(t) = \sum_{k=1}^{\infty} \rho_k g^2(t_k)\phi^2(t_k).$$

From this formula it is not difficult to deduce that the operator $g(A)$ is completely continuous.

In fact, let $\{\phi_n\}$ be some σ-bounded ($|\phi_n|_\sigma < M$, $n = 1, 2, \cdots$) sequence of elements of C_R. Then

$$M^2 > |\phi_n|_\sigma^2 = \int_{-R}^{R} \phi_n^2(t)\, d\sigma(t) \geq \sum_{k=1}^{\infty} \rho_k \phi_n^2(t_k),$$

and therefore, for any positive integer k

$$|\phi_n(t_k)| \leq \frac{M}{\sqrt{\rho_k}} \quad (n = 1, 2, \cdots).$$

Therefore using Cantor's diagonal process we may extract a subsequence $\{\phi_{n_\nu}\}$ for which there will exist the limits

$$r_k = \lim \phi_{n_\nu}(t_k) \quad (k = 1, 2, \cdots). \tag{27}$$

We now take ϵ to be an arbitrary positive number and determine $n_0(\epsilon)$ so that

$$|g(t_k)| < \frac{\epsilon}{M} \text{ for } k > n_0. \tag{28}$$

Next we determine a number $N(\epsilon)$ such that

$$\sum_{k=1}^{n_0} \rho_k g^2(t_k) |\phi_{n_\nu}(t_k) - \phi_{n_\mu}(t_k)| < \epsilon^2 \text{ for } \mu, \nu > N.$$

By (27) this is possible. Then if we again take into account (28)

$$|g(A)\phi_{n_\nu} - g(A)\phi_{n_\mu}|_\sigma^2 = \sum_{k=1}^{n_0} \rho_k g^2(t_k) |\phi_{n_\nu}(t_k) - \phi_{n_\mu}(t_k)|^2 +$$

$$+ \sum_{k=n_0+1}^{\infty} \rho_k g^2(t_k) |\phi_{n_\nu}(t_k) - \phi_{n_\mu}(t_k)|^2 <$$

$$< \epsilon^2 + \frac{\epsilon^2}{M^2} \sum_{n_0+1}^{\infty} \rho_k |\phi_{n_\nu}(t_k) - \phi_{n_\mu}(t_k)|^2 < 5\epsilon^2 \text{ for } \mu, \nu > N,$$

because

$$\sum_{k=n_0+1}^{\infty} \rho_k |\phi_{n_\nu}(t_k) - \phi_{n_\mu}(t_k)|^2 \leq \int_{-R}^{R} |\phi_{n_\nu}(t) - \phi_{n_\mu}(t)|^3 \, d\sigma(t) = |\phi_{n_\nu} - \phi_{n_\mu}|_\sigma^2 \leq$$

$$\leq \{|\phi_{n_\nu}| + |\phi_{n_\mu}|\}^2 \leq 4M^2.$$

Therefore the sequence $\{g(A)\phi_{n_\nu}\}$ is σ-convergent which means the operator $g(A)$ is completely continuous.

Assertion 2° is proved.

4. A comparison of criteria 1° and 2° leads us to the theorem:

Theorem 2. *In order for a sequence* $\{s_k\}_0^\infty$, *positive definite in the interval* $(-\infty, \infty)$, *to admit a representation*

$$s_k = \int_{-\infty}^{\infty} t^k \, d\sigma(t) \quad (k = 0, 1, 2, \cdots),$$

where the non-decreasing function $\sigma(t)$ *has a bounded set of growth points* E_σ, *with only the points* $\alpha_1, \alpha_2, \cdots, \alpha_p$ *given in advance allowable as cluster points, it is necessary and sufficient that the numbers*

$$a_i, \; b_i \quad (i = 0, 1, 2, \cdots) \tag{29}$$

form bounded sets and that

$$\lim_{i, k \to \infty} g_{ik} = 0, \tag{30}$$

where the matrix

$$\|g_{ik}\|_0^\infty = g(\|A\|) \qquad (\|A\| = \begin{Vmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_1 & a_1 & b_0 & 0 & \cdots \\ 0 & b_2 & a_2 & b_1 & \cdots \\ 0 & 0 & b_3 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix})$$

and the polynomial

$$g(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_p).$$

Proof. Indeed, as we know from Theorem 1, the set E_σ is bounded if and only if the two sets of numbers of (29) are bounded. On the other hand, in order for the set E_σ not to have cluster points distinct from the set of points $\alpha_1, \alpha_2, \cdots$ $\cdots, \alpha_p$ it is necessary and sufficient that the set $g(E_\sigma)$ where $g(x) = (x - \alpha_1) \cdots (x - \alpha_p)$ has a single cluster point which equals zero. This last condition with the boundedness of the set E_σ is equivalent (according to criterion 2°) to the condition that the operator $g(A)$ be completely continuous which is in turn equivalent (according to criterion 1°) to condition (30).

In the particular case when $p = 1$, $\alpha_1 = \alpha$ and, therefore, $g(x) = x - \alpha$, we obtain the theorem:

Theorem 3. *In order for the sequence* $\{s_k\}_0^\infty$ *positive definite in the interval* $(-\infty, \infty)$, *to admit a representation*

$$s_k = \int_{-\infty}^{\infty} t^k \, d\sigma(t) \quad (k = 0, 1, \cdots), \tag{31}$$

in which the non-decreasing function $\sigma(t)$, *has a bounded set of points of growth*

with a unique cluster point equal to α, *it is necessary and sufficient that there be satisfied the conditions*

$$\lim_{i\to\infty} a_i = \alpha, \quad \lim_{i\to\infty} b_i = 0.$$

In the particular case in which the sequence $\{s_k\}_0^\infty$ is positive definite in the interval $<0, \infty>$ and, therefore, admits a representation

$$s_k = \int_0^\infty t^k \, d\sigma(t) \quad (k = 0, 1, \cdots),$$

Theorem 3 was obtained (by entirely different methods) by Stieltjes (cf. supplement to [39a]).

As N. I. Ahiezer remarked to us, by changing Stieltjes' reasoning somewhat, one may by his method obtain the general case of Theorem 3. Still, it seems improbable to us that Theorem 2 could be proved by this method.

§4

We shall especially concentrate on the case of Theorem 3 where $\alpha = 0$. In this case let $t_1, t_2, \cdots, t_n, \cdots (t_n \to \infty)$ be a sequence of points of growth of the function $\sigma(t)$ distinct from zero. Let

$$m_0 = \sigma(+0) - \sigma(-0), \quad m_i = \sigma(t_i + 0) - \sigma(t_i - 0) \quad (i = 1, 2, \cdots). \tag{32}$$

Then by (31)

$$s_0 = m_0 + \sum_1^\infty m_k, \quad s_i = \sum_{k=1}^\infty m_k t_k^i \quad (i = 1, 2, \cdots). \tag{33}$$

It is easy to see that the series

$$F(z) = m_0 + \sum_{i=1}^\infty \frac{m_i}{1 - t_i z} \tag{34}$$

is uniformly convergent in every finite region which does not contain the points $\frac{1}{t_i}$ $(i = 1, 2, \cdots)$, and therefore $F(z)$ is a meromorphic function with the simple real poles

$$x_i = \frac{1}{t_i} \quad (i = 1, 2, \cdots) \tag{35}$$

and with the residues

$$-\mu_i = \frac{m_i}{t_i} \quad (i = 1, 2, \cdots). \tag{36}$$

Moreover, by relations (33), $F(z)$ admits the following expansion in a neighborhood of the regular point $z = 0$,

$$F(z) = s_0 + s_1 z + s_2 z^2 + \cdots .$$

Conversely, let the power series $\sum_0^\infty s_k z^k$ converge in a neighborhood of the point O and define some meromorphic function $F(z)$, which admits a representation

$$F(z) = m_0 + \sum_{i \neq 1}^{\infty} \frac{\mu_i}{x_i - z},$$

where x_i $(i = 0, 1, 2, \cdots;\ x_i \to \infty)$ are all real,

$$m_0 \geq 0 \text{ and } \operatorname{sign} \mu_i = \operatorname{sign} x_i \quad (i = 1, 2, \cdots).$$

Having defined the numbers t_i and m_i $(i = 1, 2, \cdots)$ from relations (35) and (36), we may represent the function $F(z)$ in the form (34) and hence the numbers s_i $(i = 0, 1, 2, \cdots)$ in the form (33).

Thus Theorem 3 is equivalent to the following theorem:

Theorem 4. *In order that the real power series*

$$s_0 + s_1 z + s_2 z^2 + \cdots$$

converge in some neighborhood of the point O and determine a meromorphic function $F(z)$, which admits a representation

$$F(z) = m_0 + \sum_{i=1}^{\infty} \frac{\mu_i}{x_i - z}, \tag{37}$$

where x_i $(i = 1, 2, \cdots)$ are all real,

$$m_0 \geq 0,\ \operatorname{sign} \mu_i = \operatorname{sign} x_i \quad (i = 1, 2, \cdots),$$

it is necessary and sufficient that all the determinants

$$D_n = |\, s_{i+k} \,|_0^n \quad (n = 0, 1, 2, \cdots)$$

be positive and that

$$\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i = 0. \tag{38}$$

This theorem is closely related to a theorem of Grommer [27].

Theorem 5 (Grommer). *In order that the real meromorphic function $F(z)$, regular at the point O and having there an expansion*

$$F(z) = s_0 + s_1 z + s_2 z^2 + \cdots$$

admit representation (37) *it is necessary and sufficient that all of the determinants* $D_n = |s_{i+k}|_0^n$ $(n = 0, 1, \cdots)$ *be positive.*

Thus, in Grommer's theorem it is required beforehand that the function $F(z)$ be meromorphic while in Theorem 4 the meromorphicity is a consequence of conditions (38) [of course in the presence of the conditions: $D_n > 0$ $(n = 1, 2, \cdots)$].

At the same time, Grommer's theorem is not an immediate consequence of Theorem 4, since it has not been established that under the conditions: $D_n > 0$ $(n = 0, 1, 2, \cdots)$ the meromorphicity of the function $F(z)$ implies the conditions (38).

We shall prove Grommer's theorem independently of Theorem 4.

After all we already know, the necessity of the conditions of Grommer's theorem is trivially established. Therefore, it remains to prove that a meromorphic function satisfying the conditions: $D_n > 0$ $(n = 1, 2, \cdots)$ admits representation (37). But if these conditions are fulfilled one has the equalities

$$s_k = \int_{-\infty}^{\infty} t^k \, d\sigma(t) \quad (k = 0, 1, 2, \cdots), \tag{39}$$

where $\sigma(t)$ is some non-decreasing function.

We shall show that for any $\epsilon > 0$ [1]

$$\int_{\frac{1}{\rho}+\epsilon}^{\infty} d\sigma(t) = \int_{-\infty}^{\frac{1}{\rho}+\epsilon} d\sigma(t) = 0. \tag{40}$$

Suppose to the contrary, for example, that for some $\epsilon > 0$

$$h = \int_{\frac{1}{\rho}+\epsilon}^{\infty} d\sigma(t) > 0.$$

But then

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1) We have already proved on one occasion that (39) follows from the equality $\frac{1}{\rho} = \overline{\lim_{n\to\infty}} \sqrt[n]{|s_n|}$ (cf. the proof of Theorem 11 of Chapter II of Article I); below we reiterate our earlier reasoning.

$$s_{2k} = \int_{-\infty}^{\infty} t^{2k}\, d\sigma(t) \geq \int_{\frac{1}{\rho}+\epsilon}^{\infty} t^{2k}\, d\sigma(t) \geq (\frac{1}{\rho} + \epsilon)^{2k} h,$$

and therefore,

$$\frac{1}{\rho} = \overline{\lim_{k-\infty}} \sqrt[k]{|s_k|} \geq \overline{\lim_{k\to\infty}} \sqrt[2k]{s_{2k}} \geq (\frac{1}{\rho} + \epsilon) \overline{\lim_{k\to\infty}} \sqrt[2k]{h} = \frac{1}{\rho} + \epsilon.$$

This is a contraction.

Without loss of generality we may suppose that

$$\sigma(\frac{1}{\rho} + 0) = \sigma(\frac{1}{\rho}) \text{ and } \sigma(-\frac{1}{\rho} - 0) = \sigma(-\frac{1}{\rho}).$$

Then by (40) equalities (39) may be rewritten so that:

$$s_k = \int_{-\frac{1}{\rho}}^{\frac{1}{\rho}} t^k\, d\sigma(t) \quad (k = 0, 1, 2, \cdots).$$

Thus

$$F(z) = \int_{-\frac{1}{\rho}}^{\frac{1}{\rho}} \frac{d\sigma(t)}{1 - tz} \tag{41}$$

with $|z| < \rho$, and this means for any z outside of the interval $<-\frac{1}{\rho}, \frac{1}{\rho}>$. From this equality it follows that the function $F(z)$ may only have real poles outside of the interval: $<-\frac{1}{\rho}, \frac{1}{\rho}>$; we represent them as $x_1, x_2, \cdots (x_n \to \infty)$.

Let

$$\Phi(z) = zF(\frac{1}{z}) = \int_{-\frac{1}{\rho}}^{\frac{1}{\rho}} \frac{d\sigma(t)}{t - z}.$$

It is obvious that only singularities of the single-valued real function $\Phi(z)$ will be the points

$$t_i = \frac{1}{x_i} \quad (i = 1, 2, \cdots) \quad (t_i \to 0).$$

And hence, as we know (see remark 1 to Theorem 8 of Chapter II, Article I) the only points of growth of the function $\sigma(t)$ will be the points t_i $(i = 1, 2, \cdots)$ and

perhaps the point 0.

Having therefore determined with the help of equalities (32), (35), and (36) the quantities $m_0 \geq 0$, μ_i and x_i $(i = 1, 2, \cdots)$, we obtain the required representation (37) from representation (41).

This proves Grommer's theorem. [1)]

Remark 1. From the proof of the theorem it is not difficult to recognize that the requirement that the function $F(z)$ be meromorphic can be replaced by the weaker requirement that the function be a single-valued analytic function regular everywhere on the real axis except at a denumerable set of points, and this requirement as is easily seen is essential.

Remark 2. The condition that the determinants D_n $(n = 0, 1, \cdots)$ be positive, is equivalent to the one that the forms

$$\sum_0^n s_{i+k} \xi_i \xi_k \quad (n = 0, 1, 2, \cdots) \tag{42}$$

be positive. If we relinquish the requirement that there is an infinite number of summands in representation (37), i.e., the requirement that $F(z)$ is not a rational function, then in Theorem 5 the condition of positiveness of the determinants D_n $(n = 0, 1, 2, \cdots)$ must be replaced by the non-negativeness of forms (42). It is also easy to see that if the forms (42) are singular beginning exactly with $n = p$, then in representation (37) among the quantities

$$m_0, \mu_1, \mu_2, \cdots$$

there are exactly p non-zero ones.

With the help of a proposition of Kronecker [2)] which is not at all trivial, it can be proved that the latter case occurs if and only if

---

1) A new, comparatively simple proof of Theorem 4 was recently given by N. N. Meĭman [14]. However, this proof does not apply if instead of the meromorphicity of the function $F(z)$ in Grommer's theorem we require that this function belongs to the class of functions described in note 1.

With more general assumptions which are also covered in very special cases by our remark 1, Grommer's theorem was proved by N. G. Čebotarev [17], using an elaborate apparatus.

The proof we have mentioned is very close to Grommer's own proof; it corresponds to the essence of the matter and in our opinion is simple enough.

2) Cf. Bieberbach, Lehrbuch der Functionentheorie, vol. II, p. 321, 1931.

$$D_0 > 0,\ D_1 > 0, \cdots, D_{p-1} > 0,\ D_p = D_{p+1} = \cdots = 0.$$

Since it will be more convenient from now on to use forms rather than determinants, we shall make the following convention. We shall say that an infinite real form

$$\sum_{i,k=0}^{\infty} a_{ik}\xi_i\xi_k \quad (a_{ik} = a_{ki}) \tag{43}$$

is positive (non-negative) if all its partial sums

$$\sum_{i,k=0}^{n} a_{ik}\xi_i\xi_k \quad (n = 0, 1, 2, \cdots) \tag{44}$$

are positive (non-negative). Furthermore, we shall say that the form (43) has rank p if for $n \geq p$ the forms (44) have rank of exactly p.

We shall make the same corollaries to Theorem 5 that Grommer [27] did.

Theorem 6 (Grommer). *Let the real meromorphic function $F(z)$ be regular at the point O and have there an expansion*

$$F(z) = s_0 + s_1 z + \cdots.$$

In order for the function $F(z)$ to admit a representation

$$F(z) = A_0 + A_1 z + \cdots + A_p z^p + \\ + \sum r_i \left\{ \frac{1}{x_i - z} - \frac{1}{x_i} - \frac{z}{x_i^2} - \cdots - \frac{z^{p-1}}{x_i^p} \right\},\ (p \geq 1), \tag{45}$$

where the poles x_i $(i = 1, 2, \cdots)$ are real numbers,

$$A_p \geq 0,\ \operatorname{sign} r_i = \operatorname{sign} x_i^{p-1} \quad (i = 1, 2, \cdots),$$

it is necessary and sufficient that the form

$$\sum_{i,k=0}^{\infty} s_{p+i+k}\xi_i\xi_k$$

be non-negative.

The function $F(z)$ is a rational function if and only if this form has finite rank.

Proof. Indeed, representation (45) is equivalent to

$$F_p(z) = s_p + s_{p+1}z + s_{p+2}z^2 + \cdots = A_p + \sum_{i=1}^{\infty} \frac{r_i}{x_i^p} \frac{1}{x_i - z},$$

in which

$$A_p \geq 0 \text{ and } \operatorname{sign} \frac{r_i}{x_i^p} = \operatorname{sign} x_i \quad (i = 1, 2, \cdots).$$

Hence, if we apply Theorem 5 (with remark 2) to the meromorphic function $F_p(z)$ we get Theorem 6.

Theorem 7 (Grommer). *In order for a real entire transcendental function $G(z)$ to be of genus $\leq p$, to within a factor $e^{-az^{p+1}}$ $(a \geq 0)$, and have only real roots for p even and only positive roots for p odd it is necessary and sufficient that the form*

$$\sum_{i,k=0}^{\infty} s_{p+i+k} \xi_i \xi_k,$$

in which the coefficients s_k are determined from the expansion

$$-\frac{G'(z)}{G(z)} = s_0 + s_1 z + \cdots$$

be non-negative.

Moreover $G(z)$ has an infinite set of roots if and only if the form is positive.

Proof. Indeed, from

$$G(z) = \pm e^{a_0 + a_1 z + \cdots + a_{p+1} z^{p+1}} \prod \left(1 - \frac{z}{x_i}\right) e^{\frac{z}{x_i} + \cdots + \frac{z^p}{p x_i^p}}$$

it follows that

$$-\frac{G'(z)}{G(z)} = a_1 + 2a_2 z + \cdots + (p+1) a_{p+1} z^p + \sum \left\{ \frac{1}{z - x_i} + \frac{1}{x_i} + \cdots + \frac{z^{p-1}}{x_i^{p-1}} \right\}. \tag{46}$$

If the conditions $a_{p+1} \geq 0$, x_i $(i = 1, 2, \cdots)$ real for even p (positive for odd p) hold, then in this case and only this case, expression (46) is a meromorphic function of the type discussed in Theorem 6. From this Theorem 7 follows.

Whatever the entire function $G(z)$ of finite genus may be, by selecting an arbitrary even (or odd) number p not less than the genus of the function $G(z)$ we may always check with the help of Theorem 7 whether all of the roots of the function are real (or positive).

N. G. Čebotarev solved a more general question. He demonstrated a simple

and ingenious method which makes it possible to determine how many real or positive roots of the entire function $G(z)$ there are among the first m in modulus, independent of whether the genus of this function was finite or infinite (cf. §1, 2 of his article [17]).

§5

Let $g(z)$ and $h(z)$ be two entire functions with real coefficients and let

$$F(z) = \frac{g(z)}{h(z)} = s_0 + s_1 z + s_2 z^2 + \cdots .$$

As we know from Theorem 6, the expansion

$$F(z) = \frac{g(z)}{h(z)} = A_0 + A_1 z + \sum \left\{ \frac{r_i}{x_i - z} - \frac{r_i}{x_i} \right\}, \tag{47}$$

where $x_i \neq 0$ $(i = 0, 1, 2, \cdots)$ are all real, $A_0 \gtrless 0$, $A_1 \geq 0$ and $r_i > 0$ $(i = 1, 2, \cdots)$ holds if and only if the form

$$S(g, h, \xi_0, \xi_1, \cdots) = \sum_0^\infty s_{i+k} \xi_i \xi_k \tag{48}$$

is non-negative.

In this section we shall study entire complex functions $f(z) = g(z) - ih(z)$, which give rise to complex functions $F = g/h$ for which expansion (47) is valid. For this we first prove the following proposition:

Theorem 8. *In order that the real meromorphic function $F(z)$ regular at the point O* [1] *admit an expansion* (47), *in other words, in order that the form S be non-negative it is necessary and sufficient that the function $F(z)$ assume in the upper half-plane $\Im z \geq 0$ values which lie in that half-plane.* [2]

Proof. The necessity of the condition follows from the fact that if the function $F(z)$ admits representation (47), then

$$\Im F(z) = \left[A_1 + \sum \frac{r_i}{|x_i - z|^2} \right] y.$$

---

1) If we delete the requirement that the function $F(z)$ is regular at O then the theorem retains its sense if in expansion (47) it is not required that $x_i \neq 0$ $(i = 1, 2, \cdots)$.

2) N. G. Čebotarev [17] also used this theorem. As we shall see, it is a very specialized consequence of theorems already established by Nevanlinna and others (cf. Chapter II, Article I).

We shall prove the sufficiency of the condition. Supposing that the condition is satisfied, we shall examine the function $F(\frac{1}{z})$. It is real on the real axis, regular outside of some segment $<-R, R>$ and in the upper half-plane assumes values lying in the lower half-plane. Therefore (cf. Theorem 11, Chapter II, Article I) it admits the representation

$$F(\frac{1}{z}) = \int_{-R}^{R} \frac{d\sigma(t)}{z-t} + F(0), \tag{49}$$

where $\sigma(t)$ $(-R \leq t \leq R)$ is a non-decreasing function. Hence

$$s_{k+1} = \int_{-R}^{R} t^k \, d\sigma(t) \quad (k = 0, 1, \cdots),$$

and therefore form (48) is non-negative. The theorem is proved.

We note that expansion (47) may be immediately obtained from representation (49) [without recourse to form (48)] by reasoning analogously to the proof of Theorem 5.

We now prove a proposition which will be generalized considerably later.

Theorem 9. *Let*

$$f(z) = g(z) - ih(z) \quad (h(0) \neq 0)$$

be a polynomial or an entire transcendental function of null genus with the real functions $g(z)$ and $h(z)$ not having common zeros.

In order that all of the zeros of the function $f(z)$ be located within the upper half-plane $\Im z \geq 0$ it is necessary and sufficient that the form

$$S(g, h; \xi_0, \xi_1, \cdots) = \sum_{0}^{\infty} s_{i+k+1} \xi_i \xi_k$$

be non-negative.

If the form S[1] *is positive, then the function $f(z)$ has an infinite set of zeros lying within the upper half-plane; if the non-negative form S has a finite rank p, then the function $f(z)$ has exactly p zeros in the upper half-plane, counting them with their multiplicities.*

Proof. Indeed, let

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1) Later on we shall replace the form S with another form after which we shall be free from the restriction $h(0) \neq 0$.

$$f(z) = c \prod \left(1 - \frac{z}{z_k}\right),$$

where

$$\Im z_k > 0 \quad (k = 1, 2, \cdots).$$

Then

$$\Phi(z) = \frac{f(z)}{\overline{f}(z)} = \frac{c}{\overline{c}} \prod \frac{z_k - z}{\overline{z}_k - z} \frac{\overline{z}_k}{z_k}$$

and since

$$\left| \frac{z_k - z}{\overline{z}_k - z} \right| \leq 1 \text{ for } \Im z \geq 0,$$

then

$$|\Phi(z)| \leq 1 \text{ for } \Im z \geq 0.$$

Bearing in mind that the function

$$w = \frac{1}{i} \frac{\zeta + 1}{\zeta - 1}$$

transforms the unit circle $|\zeta| \leq 1$ into the upper half-plane, we conclude that the function

$$F(z) = \frac{g(z)}{h(z)} = \frac{1}{i} \frac{\Phi + 1}{\Phi - 1}$$

assumes in the upper half-plane values which lie in this half-plane. Hence by Theorem 8 the form S is non-negative.

Conversely, let the form S be non-negative; then according to Theorem 8 the function $F(z) = g/h$ takes on in the lower half-plane values belonging to the same half-plane and therefore the roots of the equation

$$\frac{g(z)}{h(z)} = i \quad (f(z) = g(z) - ih(z) = 0)$$

all lie in the upper half-plane.

It remains to clarify the second part of the theorem.

If the form S is positive, then an infinite number of summands figure in expansion (47) and therefore $f(z)$ is an entire transcendental function with an infinite number of zeros.

If, on the other hand, the form S is singular of rank p then in the equalities

$$s_{k+1} = A_1 + \Sigma\, r_j \left(\frac{1}{x_j}\right)^{k+2} \quad (k = 0,\ 1,\ 2, \cdots)$$

of the quantities

$$A_1 \geq 0,\ \ r_1 > 0,\ \ r_2 > 0, \cdots \tag{50}$$

only p are different from zero and therefore by (47) $f(z)$ is a polynomial of degree p.

This proves the theorem.

Naturally the question arises: what may be said about entire functions $f(z)$ to which there correspond non-negative forms S and upon which no other restrictions are imposed?

It is not difficult to answer the question in the case in which the form S is non-negative and of finite rank; in fact one has

Theorem 10. *Let*

$$f(z) = g(z) - ih(z) \quad (h(0) \neq 0)$$

be some entire function. In order that this function admit a representation

$$f(z) = e^{i\theta} E(z) \prod_{i=1}^{p} \left(1 - \frac{z}{z_i}\right), \tag{51}$$

where $E(z)$ is an entire real function and z_i $(i = 1, 2, \cdots, p)$ are points lying in the upper half-plane $\Im z > 0$, it is necessary and sufficient that the form

$$S(g, h; \xi_0, \xi_1, \cdots) = \sum_{0}^{\infty} s_{j+k} \xi_j \xi_k$$

be non-negative and have rank equal to p.

Proof. Let the function $f(z)$ admit representation (51). Let

$$f_1(z) = g_1(z) - ih_1(z) = e^{i\theta} \prod_{k=1}^{p} \left(1 - \frac{z}{z_k}\right); \tag{52}$$

then obviously,

$$g(z) = E(z)\, g_1(z) \text{ and } h(z) = E(z)\, h_1(z),$$

and therefore

$$\frac{g(z)}{h(z)} = \frac{g_1(z)}{h_1(z)} = s_0 + s_1 z + \cdots.$$

Applying Theorem 9 to the function $f_1(z)$, we find that the form S has the required properties.

Now, conversely, let the form S be non-negative and have rank $= p$. Then the function $F(z) = g/h$ admits expansion (47), in which among the quantities (50) only p are different from zero, and therefore this is a real rational function the exact degree of which equals p.

Let

$$F(z) = \frac{g(z)}{h(z)} = \frac{g_1(z)}{h_1(z)}, \tag{53}$$

where $g_1(z)$ and $h_1(z)$ are polynomials of degree $\leq p$ (and one of them is of exactly degree p). Since the function $F(z)$ takes on in the open upper (lower) half-plane values which lie in the same half-plane, the roots of the polynomial $f_1(z) = g_1(z) - ih_1(z)$ lie within the upper half-plane. Therefore, the polynomial $f_1(z)$ admits representation (52). On the other hand, it is apparent from (53) that every root of the function $g_1(z)$ [the function $h_1(z)$] is a root of the function $g(z)$ [the function $h(z)$] and moreover of not less multiplicity. Therefore,

$$E(z) = \frac{g(z)}{g_1(z)} = \frac{h(z)}{h_1(z)}$$

is an entire real function. From this there follows representation (51).

2. For the study of arbitrary entire functions to which there correspond positive forms S a series of lemmas is necessary:

Lemma 1. *Let $G(z)$ be a real entire function which satisfies the condition*

$$\Im G(z) \geq 0 \text{ for } \Im z \geq 0.$$

Then $G(z) = az + b$, where $a \geq 0$, $b \geqq 0$. [1)]

Proof. Indeed, by Theorem 8 of Chapter II of Article I the function $G(z)$ admits a representation

$$G(z) = az + b + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z}\, d\sigma(t),$$

where $a \geq 0$, $b \geqq 0$, and $\sigma(t)$ $(-\infty < t < \infty)$ is a non-decreasing function. Since the function $G(z)$ is regular and real on the real axis, then by the remark to the Theorem 8 cited the function $\sigma(t) \equiv$ const., which proves the lemma.

Lemma 2. *Let the meromorphic function $\Phi(z)$ satisfy the conditions*

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1) This lemma was used for a different purpose by N. G. Čebotarev [17]; in his article there is mentioned an immediate elementary proof of the lemma.

$$|\Phi(z)| \begin{matrix} \leq 1 \text{ for } \Im z \geq 0 \\ \geq 1 \text{ for } \Im z \leq 0. \end{matrix} \tag{54}$$

Then $\Phi(z)$ *admits a representation*

$$\Phi(z) = e^{-(az+b)i} \prod \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\bar{z}_k}}, \tag{55}$$

where $a \geq 0$, $b \leqq 0$ *and*

$$\Im z_k > 0 \quad (k = 1, 2, \cdots). \tag{56}$$

If the number of zeros z_k $(k = 1, 2, \cdots)$ *is infinite then the conditions*

$$z_k \to \infty \text{ and } \sum_{k=1}^{\infty} \Im \left[-\frac{1}{z_k} \right] < \infty \tag{57}$$

are satisfied, which insures the convergence of the product (55) *for* $z \neq \bar{z}_k$ $(k = 1, 2, \cdots)$.

Proof. By virtue of (54), all the zeros of the function $\Phi(z)$ lie within the upper half-plane $\Im z > 0$; let z_k $(k = 1, 2, \cdots, n;\ \Im z_k > 0)$ be some zeros of the function $\Phi(z)$. Let

$$\Phi_n(z) = \Phi(z) \prod_{k=1}^{n} \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\bar{z}_k}}. \tag{58}$$

By (54) and the fact that

$$\prod_{k=1}^{n} \left| \frac{z_k - z}{\bar{z}_k - z} \right| = 1 \text{ for } \Im z = 0,$$

by the maximum modulus principle it follows that

$$|\Phi_n(z)| \leq 1 \text{ for } \Im z \geq 0. \tag{59}$$

Since $|\Phi(z)| = 1$ for $\Im z = 0$, then by the Riemann–Schwarz principle the zeros and poles of the function $\Phi(z)$ are symmetrically located relative to the real axis. Therefore, if $\Phi(z)$ has no other zeros besides z_k $(k = 1, 2, \cdots, n)$, then $\Phi_n(z)$ is an entire function. Therefore, in this case

$$\Phi_n(z) = e^{-iG(z)}, \tag{60}$$

where $G(z)$ is entire and moreover a real function ($|\Phi_n(z)| = 1$ for $\Im z = 0$). Furthermore, by virtue of (59)

$$\Im G(z) \geq 0 \text{ for } \Im z \geq 0.$$

Therefore, according to Lemma 1

$$G(z) = az + b, \tag{61}$$

where $a \geq 0$, $b \gtreqless 0$.

Combining (61), (60), and (58) we arrive at representation (55).

We shall now consider the other case when $\Phi(z)$ has an infinite number of zeros. Let $\{z_k\}_1^\infty$ be a sequence of the zéros of the function $\Phi(z)$ in which each zero appears as many times as its multiplicity.

From inequality (59) it follows that

$$|\Phi(z)| \leq \prod_{k=1}^{n} \left| \frac{z_k - z}{\bar{z}_k - z} \right|. \tag{62}$$

Let $\Im z > 0$ and $z \neq z_k$ $(k = 1, 2, \cdots)$. From (62) we conclude that for such z the infinite product

$$\prod_{k=1}^{\infty} \left| \frac{z_k - z}{\bar{z}_k - z} \right|,$$

and this means also the infinite product

$$\prod_{k=1}^{\infty} \left| \frac{z_k - z}{\bar{z}_k - z} \right|^2$$

converges. However, the last product converges if and only if the following series converges:

$$\sum_{k=1}^{\infty} \left(1 - \left| \frac{z_k - z}{\bar{z}_k - z} \right|^2 \right) = 4 \Im z \sum_{k=1}^{\infty} \frac{\Im z_k}{|\bar{z}_k - z|^2}. \tag{63}$$

Since $z_k \longrightarrow \infty$, then together with (63) the following series is also convergent

$$\sum_{k=1}^{\infty} \frac{\Im z_k}{|z_k|^2} = \sum_{k=1}^{\infty} \Im \left(-\frac{1}{z_k} \right). \tag{64}$$

But from this follows the uniform convergence of the infinite product

$$\prod_{k=1}^{\infty} \frac{1-\frac{z}{z_k}}{1-\frac{z}{\overline{z}_k}} \tag{65}$$

in every finite closed region D of the plane which does not include the points $\overline{z}_k$ $(k = 1, 2, \cdots)$.

In fact,

$$|u_k(z)| = \left| 1 - \frac{1-\frac{z}{z_k}}{1-\frac{z}{\overline{z}_k}} \right| = \frac{\Im z_k |z|}{|\overline{z}_k - z|\,|z_k|} \leq C_D \frac{\Im z_k}{|z_k|^2} \quad (k = 1, 2, \cdots),$$

where

$$C_D = \sup \frac{|z|\,|z_k|}{|\overline{z}_k - z|} \quad (z \subset D, \ k = 1, 2, \cdots),$$

and therefore in the region D the series $\Sigma_1^{\infty} u_k(z)$ is majorized by the convergent series (64), the members of which are multiplied by the constant C_D.

Let

$$\Phi_{\infty}(z) = \lim_{n \to \infty} \Phi_n(z) = \Phi(z) \prod_{k=1}^{\infty} \frac{1-\frac{z}{z_k}}{1-\frac{z}{\overline{z}_k}}. \tag{66}$$

By virtue of (59)

$$|\Phi_{\infty}(z)| \leq 1 \text{ for } \Im z \geq 0. \tag{67}$$

Furthermore, $\Phi_{\infty}(z)$ is an entire real function; therefore by the arguments we have already outlined, it follows from (67) that

$$\Phi_{\infty}(z) = e^{-(az+b)i}, \tag{68}$$

where $a \geq 0$, $b \gtrless 0$.

Combining (68) and (66) we again come to representation (55). The lemma is proved.

Lemma 3. *If for a sequence of points* $\{z_k\}$ *conditions* (56) *and* (57) *of Lemma 2 are satisfied, then the series*

$$\frac{1}{p} \sum_{k=p}^{\infty} \Im \left[\frac{1}{z_k^p} \right] \quad (p = 1, 2, \cdots) \tag{69}$$

as well as the double series (for any complex z)

$$\sum_{p=1}^{\infty} \sum_{k=p}^{\infty} \Im\left(\frac{1}{z_k^p}\right) \frac{z^p}{p} \tag{70}$$

is absolutely convergent.

Proof. Indeed, for every complex z

$$|\Im z^p| = \left|\frac{z^p - \bar{z}^p}{2}\right| = \left|\frac{z - \bar{z}}{2}\right| \left|\sum_{k=0}^{p-1} z^k \bar{z}^{p-1-k}\right| \leq p\,|\Im z|\,|z|^{p-1},$$

and therefore,

$$\left|\Im\left(\frac{1}{z^p}\right)\right| = \frac{|\Im z^p|}{|z|^{2p}} \leq p\,\frac{|\Im z|}{|z|^{p+1}}.$$

By conditions (57) we conclude from this that all series in (69) are absolutely convergent and more than that

$$\sum_{p=1}^{\infty} \sum_{k=p}^{\infty} \left|\Im\left(\frac{1}{z_k^p}\right)\right| \frac{|z|^p}{p} \leq \sum_{p=1}^{\infty} \sum_{k=p}^{\infty} \frac{\Im z_k}{|z_k|^{p+1}} |z|^p \leq$$

$$\leq \sum_{p=1}^{\infty} \frac{|z|^p}{\rho_p^{p-1}} \sum_{k=p}^{\infty} \frac{\Im z_k}{|z_k|^2} \leq \left(\sum_{k=1}^{\infty} \frac{\Im z_k}{|z_k|^2}\right)\left(\sum_{p=1}^{\infty} \frac{|z|^p}{\rho_p^{p-1}}\right),$$

where

$$\rho_p = \inf(|z_p|, |z_{p+1}|, \cdots).$$

Since $\rho_p \to \infty$, then the series

$$\sum_{p=1}^{\infty} \frac{|z|^p}{\rho_p^{p-1}}$$

converges for any z and therefore the lemma is proved.

We shall denote the sum of the series (69) by γ_p ($p = 1, 2, \cdots$) and the sum of the series (70) by $\Gamma(z)$; then

$$\Gamma(z) = \sum_{p=1}^{\infty} \gamma_p z^p,$$

where

$$\gamma_p = \frac{1}{p} \sum_{k=p}^{\infty} \Im \left[\frac{1}{z_k^p} \right] \quad (p = 1, 2, \cdots).$$

Lemma 4. *If the sequence of points* $\{z_k\}$ *satisfies conditions* (56) *and* (57) *of Lemma* 2, *then the entire function*

$$Q(z) = e^{-i\Gamma(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{\frac{z}{z_k} + \frac{1}{2}\left(\frac{z}{z_k}\right)^2 + \cdots + \frac{1}{k}\left(\frac{z}{z_k}\right)^k} \tag{71}$$

has the property that

$$\frac{Q(z)}{\overline{Q}(z)} = \prod_{k=1}^{\infty} \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\overline{z}_k}}. \tag{72}$$

Proof. Indeed, from (71) it follows that

$$\frac{Q(z)}{\overline{Q}(z)} = e^{-2i\Gamma(z)} \prod_{k=1}^{\infty} \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\overline{z}_k}} e^{2i\left[\Im\left(\frac{1}{z_k}\right) z + \frac{1}{2}\Im\left(\frac{1}{z_k^2}\right) z^2 + \cdots + \frac{1}{k}\Im\left(\frac{1}{z_k^k}\right) z^k\right]}.$$

Bearing in mind the absolute convergence of series (70), we may easily pass from this equality to equality (72). [1]

We shall say that an entire function $H(z)$ belongs to the type (H_0), if it admits a representation

$$H(z) = e^{-i(az+b)} \prod_{k=1}^{m} \left(1 - \frac{z}{z_k}\right), \tag{73}$$

where $a > 0$, [2] $b \gtreqless 0$ and $\Im z_k > 0$ $(k = 1, 2, \cdots, m)$.

---

1) From formula (72) it also follows that if the sequence $\{z_k\}$ satisfies conditions (56) and (57) of Lemma 2, then the product (65) converges for any $z \neq \overline{z}_k$ $(k = 1, 2, \cdots)$ and represents a meromorphic function with poles $\overline{z}_k$ $(k = 1, 2, \cdots)$. It is not difficult to demonstrate that a more general proposition is true, that is: if the points z_k $(k = 1, 2, \cdots)$ lie in the open half-plane $\Im z > 0$ and do not have cluster points there, then under the condition that the product (65) converges for one non-real point the product converges and is regular at every point $z \neq \overline{z}_k (k = 1, 2, \cdots)$, which is not a cluster point of the sequence $\{z_k\}$. Furthermore, if the point O is not a cluster point of the sequence $\{z_k\}$, then the product (65) converges for $\Im z > 0$ if and only if the series $\sum_1^{\infty} \Im(-\frac{1}{z_k})$ converges. These facts will not be needed in what follows.

2) We draw the reader's attention to the fact that we require here that $a > 0$ and not $a \geq 0$.

Next, we shall say the entire function $H(z)$ belongs to type (H_p) $(p=1, 2, \cdots)$, if it is of genus $p-1$ and admits a representation

$$H(z) = \tag{74}$$

$$= e^{-i(az+b)} e^{-i(c_1 z + \cdots + c_{p-1} z^{p-1})} \prod_{k=1}^{\infty} \left(1-\frac{z}{z_k}\right) e^{\frac{z}{z_k}+\frac{1}{2}\left(\frac{z}{z_k}\right)^2+\cdots+\frac{1}{p-1}\left(\frac{z}{z_k}\right)^{p-1}},$$

where $a \geq 0$, $b \gtrless 0$,

$$\Im z_k > 0 \quad (k = 1, 2, \cdots), \quad \sum_{k=1}^{\infty} \Im\left[-\frac{1}{z_k}\right] < \infty$$

and

$$c_j = \sum_{k=1}^{\infty} \Im\left[\frac{1}{z_k^j}\right] \quad (j = 1, 2, \cdots, p-1).$$

Finally, we shall say that an entire function belongs to type (H_∞) if it is of infinite genus and admits a representation

$$H(z) = e^{-i(az+b)} Q(z), \tag{75}$$

where $a \geq 0$, $b \gtrless 0$, and $Q(z)$ is some entire function determined from formula (71).

After Lemma 4 it is not hard to see that if the function $H(z)$ belongs to one of the types (H_p) $(p = 0, 1, 2, \cdots, \infty)$, then

$$\frac{H(z)}{\overline{H}(z)} = e^{-2(az+b)i} \prod \frac{1-\frac{z}{z_k}}{1-\frac{z}{\overline{z}_k}}, \tag{76}$$

where the product written contains a finite number of factors if the type number $p = 0$ and an infinite number otherwise.

Without any effort we can now answer the question raised earlier about the full nature of an entire function to which correspond non-negative form S. The answer is formulated as the following theorem:

Theorem 11. *Let*

$$f(z) = g(z) - ih(z) \quad (h(0) \neq 0)$$

be some entire function. In order for this function to admit a representation

$$f(z) = E(z) H(z), \tag{77}$$

where $E(z)$ is an arbitrary real function and $H(z)$ is an entire function of one of the types H_p ($p = 0, 1, 2, \cdots, \infty$), it is necessary and sufficient that the form

$$S(g, h; \xi_0, \xi_1, \cdots) = \sum_0^\infty s_{j+k}\xi_j\xi_k$$

be positive.

Proof. Let $f(z)$ admit representation (77). Then by (76)

$$\Phi(z) = \frac{f(x)}{\overline{f}(x)} = e^{-2i(az+b)} \prod \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\overline{z}_k}},$$

where $a \geq 0$, $b \gtrless 0$ and $\Im z_k > 0$ $(k = 1, 2, \cdots)$; and therefore, $|\Phi(z)| \leq 1$ for $\Im z \geq 0$. Hence the meromorphic function

$$\frac{g(z)}{h(z)} = \frac{1}{i}\frac{\Phi + 1}{\Phi - 1} = s_0 + s_1 z + \cdots$$

takes on in the upper half-plane values which lie in that half-plane and therefore the form S is non-negative. Furthermore this form is positive since otherwise the function $f(z)$ would admit representation (51) which is inconsistent with representation (77).

Conversely, now let the form S be positive. Then by Theorem 8 the meromorphic function $F = g/h$ satisfies the condition: $\Im F(z) \geq 0$ or ≤ 0 depending on whether $\Im z \geq 0$ or ≤ 0.

Therefore the meromorphic function

$$\Phi(z) = \frac{f(z)}{\overline{f}(z)} = \frac{F(z) - i}{F(z) + i}$$

satisfies the conditions of Lemma 2 and hence admits representation (55). Starting with this representation we construct according to formulas (73), (74), or (75) a function $H(z)$ of one of the types (H_p) $(p = 0, 1, \cdots, \infty)$ such that

$$\frac{H(z)}{\overline{H}(z)} = e^{-i(az+b)} \prod \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\overline{z}_k}} = \frac{f(z)}{\overline{f}(z)}. \tag{78}$$

Obviously, every zero of $H(z)$ is a zero of $f(z)$ and it is moreover of at least as high multiplicity; therefore,

$$E(z) = \frac{f(z)}{H(z)}$$

is an entire function. Furthermore, by virtue of (78) $E(z) = \bar{E}(z)$, i.e., $E(z)$ is a real function. The theorem is proved.

§6

As we shall show in this section the forms $S(g, h; \xi_0, \xi_1, \cdots)$ are equivalent to other forms $B(g, h; \xi_0, \xi_1, \cdots)$, whose coefficient matrices and minors are comparatively simple to calculate. The latter forms are also more convenient because they make sense for every power series $g(z)$ and $h(z)$ without the restriction $h(0) \neq 0$. They also allow us to study in more detail the entire functions which are analogues of Hurwitz's polynomials, i.e., those polynomials which have all of their roots lying in the left half-plane.

1. Let

$$g(z) = a_0 + a_1 z + \cdots \text{ and } h(z) = b_0 + b_1 z + \cdots$$

be two power series (convergent or divergent). We shall compose the "generating function"

$$B(g, h) = \frac{g(z)h(u) - g(u)h(z)}{z - u} = \sum_{p, q=0}^{\infty} a_{pq} z^p u^q,$$

where the left member is a formal factorization in powers of z and u of the right member. If we set

$$(k, l) = a_k b_l - a_l b_k,$$

then, as is easily seen,

$$a_{pq} = a_{qp} = (p + 1, q) + (p + 2, q - 1) + \cdots + (p + q + 1, 0) \quad (p \geq q).$$

By *bézoutiant* of the series of $g(z)$ and $h(z)$ we shall mean the infinite form

$$B(g, h; \xi_0, \xi_1, \cdots) = \sum_{p, q=0}^{\infty} a_{pq} \xi_p \xi_q.$$

Furthermore, we introduce the notation

$$B(g, h; \xi_0, \cdots, \xi_{n-1}) = \sum_{p, q=0}^{n-1} a_{pq} \xi_p \eta_q \quad (n = 1, 2, \cdots).$$

It is obvious that

$$B(g, h; \xi_0, \xi_1, \cdots) = -B(-g, h; \xi_0, \xi_1, \cdots) =$$

$$= -B(g, -h; \xi_0, \xi_1, \cdots) = -B(h, g; \xi_0, \xi_1, \cdots).$$

Now let

$$g_1(z) = \phi(z)\, g(z) \text{ and } h_1(z) = \phi(z)\, h(z),$$

where

$$\phi(z) = c_0 + c_1 z + \cdots \quad (c_0 \neq 0)$$

is an arbitrary power series. We assert that

$$B(g_1, h_1; \xi_0, \xi_1, \cdots, \xi_n) = B(g, h; \eta_0, \eta_1, \cdots, \eta_n),$$

where

$$\eta_i = c_0 \xi_i + c_1 \xi_{i+1} + \cdots + c_{n-1} \xi_n \quad (i = 0, 1, \cdots, n). \tag{79}$$

Indeed,

$$B(g_1, h_1) = \phi(z)\phi(u) \frac{g(z)h(u) - h(z)g(u)}{z - u} = \phi(z)\phi(u)\, B(g, h) =$$

$$= \sum_{i,k=0}^{\infty} a_{ik} (c_0 z^i + c_1 z^{i+1} + \cdots)(c_0 u^k + c_1 u^{k+1} + \cdots),$$

and therefore,

$$B(g_1, h_1; \xi_0, \xi_1, \cdots, \xi_n) = \sum_{i,k=0}^{n} a_{ik} (c_0 \xi_i + c_1 \xi_{i+1} + \cdots +$$

$$+ c_{n-i} \xi_n)(c_0 \eta_k + \cdots + c_{n-k} \eta_k).$$

We note that the determinant of transformation (79) equals $c_0^{n+1} \neq 0$. Assuming $b_0 = h(0) \neq 0$, we set

$$\phi(z) = \frac{1}{h(z)} = c_0 + c_1 z + \cdots \quad \left(c_0 = \frac{1}{b_0} \neq 0\right),$$

then $h_1(z) \equiv 1$, but

$$g_1(z) = \frac{g(z)}{h(z)} = s_0 + s_1 z + s_2 z^2 + \cdots \tag{80}$$

and hence,

$$B(g_1, h_1) = \frac{g_1(z) - g_1(u)}{z - u} = \sum_{i,k=0}^{\infty} s_{i+k+1} z^i u^k.$$

Recalling definition (48) of the form S we thus arrive at the identity. [1)]

$$S(g, h; \xi_0, \xi_1, \cdots, \xi_n) = B(g, h; \eta_0, \eta_1, \cdots, \eta_n) \quad (n = 0, 1, \cdots), \tag{81}$$

where η_i and ξ_i are connected by the non-singular transformations (79).

Thus in particular

$$\left| a_{ik} \right|_0^n = c_0^{-2n-2} \left| s_{i+k+1} \right|_0^n = b_0^{2n+2} \left| s_{i+k+1} \right|_0^n. \tag{82}$$

By the formal rules of division it follows from (80) that

$$a_k = b_k s_0 + b_{k-1} s_1 + \cdots + b_0 s_k \quad (k = 0, 1, \cdots),$$

and hence there follows Hurwitz's identity

$$\begin{Vmatrix} b_0 & b_1 & \cdots & b_n & b_{n+1} & \cdots & b_{2n+1} \\ 0 & b_0 & \cdots & b_{n-1} & b_n & \cdots & b_{2n} \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_{n+1} \\ a_0 & a_1 & \cdots & a_n & a_{n+1} & \cdots & a_{2n+1} \\ 0 & a_0 & \cdots & a_{n-1} & a_n & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_{n+1} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ s_0 & s_1 & \cdots & s_n & s_{n+1} & \cdots & s_{2n+1} \\ 0 & s_0 & \cdots & s_{n-1} & s_n & \cdots & s_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & s_0 & s_1 & \cdots & s_{n+1} \end{Vmatrix} . B, \tag{83}$$

where the matrix

$$B = \begin{vmatrix} b_0 & b_1 & \cdots & b_{2n+1} \\ 0 & b_0 & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_0 \end{vmatrix}.$$

Equating the determinants of the left and right sides in (83) and using (82) we obtain

---

1) The left side of this equality denotes the appropriate partial sum of the infinite form $S(g, h; \xi_0, \xi_1, \cdots)$.

$$\left| a_{ik} \right|_0^n = \begin{vmatrix} b_0 & b_1 \cdots & b_{2n+1} \\ a_0 & a_1 \cdots & a_{2n+1} \\ 0 & b_0 \cdots & b_{2n} \\ 0 & a_0 \cdots & a_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot & \cdot \\ \cdot & \cdot \cdot \cdot \cdot \cdot & \cdot \end{vmatrix} \qquad (n = 0, 1, 2, \ldots). \tag{84}$$

These equalities were derived with the assumption that $b_0 \neq 0$; from continuity considerations it follows that they remain true for $b_0 = 0$.

From (81) it follows that in all of the propositions of the previous paragraph we could replace the form S with the bézoutiant $B(g, h; \xi_0, \xi_1, \cdots)$. Thus, Theorems 10 and 11 are equivalent to the following:

Theorem 10′. *Let*

$$f(z) = g(z) - ih(z) \quad (f(0) \neq 0)$$

be some entire function. In order for this function to admit a representation

$$f(z) = e^{i\theta} E(z) \prod_{k=1}^{p} \left[1 - \frac{z}{z_k} \right],$$

where $E(z)$ is an entire real function and z_i $(i = 1, 2, \cdots, p)$ are points lying in the upper half-plane $\Im z > 0$, it is necessary and sufficient that the bézoutiant $B(g, h; \xi_0, \xi_1, \cdots)$ be non-negative and have rank equal to p. [1)]

---

1) We note that if $f(z)$ is a polynomial of degree n then the variables $\xi_n, \xi_{n+1}, \cdots$ etc., do not appear in the bézoutiant $B(g, h; \xi_0, \xi_1, \cdots)$ and it reduces to a form of order n

$$B(g, h; \xi_0, \xi_1, \cdots, \xi_{n-1}). \tag{*}$$

If π is the number of positive and ν the number of negative squares of this form, then $f(z)$ has $n - \pi - \nu$ roots in common with $\overline{f}(z)$ and furthermore, it has in addition π roots in the upper half-plane and ν roots in the lower half-plane. This was demonstrated by Hermite for the case where $n = \pi + \nu$ (the form is non-singular). The following proposition may also be traced back to Sylvester and Hermite: "In order for the roots of the polynomials $g(z)$ and $h(z)$ to be real and alternate it is necessary and sufficient that the form (*) be definite." (Concerning applications of quadratic forms to the theory of determination of roots of algebraic equations see the paper by M. Naĭmark and the author [9]).

We note also that in the case when $g(z)$ and $h(z)$ are entire transcendental functions the bézoutiant $B(g, h; \xi_0, \xi_1, \cdots)$ having constant sign implies the following: "If one deletes the common zeros of the functions $g(z)$ and $h(z)$ then the remaining zeros of the functions $g(z)$ and $h(z)$ are real and alternate." This is easily seen with the aid of expansion (47).

Theorem 11′. *Let*

$$f(z) = g(z) - ih(z) \quad (f(0) \neq 0)$$

be some entire function. In order for this function to admit the representation

$$f(z) = F(z)\,H(z), \tag{85}$$

where $E(z)$ is an arbitrary real function and $H(z)$ is an entire function of one of the types (H_p) $(p = 0, 1, 2, \cdots, \infty)$, it is necessary and sufficient that the bézoutiant $B(g, h; \xi_0, \xi_1, \cdots)$ be positive.

In Theorems 10 and 11 we required that $h(0) \neq 0$; here we require that $f(0) \neq 0$. This is admissible by the following considerations. If $f(0) \neq 0$ and $h(0) = 0$ then $g(0) \neq 0$. We now consider instead of the function f the function if, where since $if = h + ig$ the roles previously performed by the functions $g(z)$ and $h(z)$ will be played by the functions $h(z)$ and $-g(z)$. One may now compose the form $S(h, -g; \xi_0, \xi_1, \cdots)$ but it is equivalent to the form $B(h, -g; \xi_0, \xi_1, \cdots) = B(g, h; \xi_0, \xi_1, \cdots)$.

From representation (85) it follows that $f(z)$ may have real roots as well as roots which lie within the upper and lower half-planes. However, to every root of $f(z)$ which lies in the lower half-plane there is always a complex conjugate root which lies in the upper half-plane and occurs with multiplicity at least as great.

Fujiwara dealt with the entire functions $f(z) = g(z) - ih(z)$ of null genus; by unrigorous arguments which are apparently difficult to correct he reached the erroneous conclusion that all of the roots of such a function lie in the upper half-plane.

2. Already in 1895 Hurwitz had proved the following proposition:

"In order that the roots of the real polynomial

$$f(z) = c_0 + c_1 z + \cdots + c_n z^n \quad (c_0 > 0)$$

all lie in the left half-plane $\Re z < 0$ it is necessary and sufficient that the n successive principal minors of the matrix

$$\begin{Vmatrix} c_1 & c_0 & 0 & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 & 0 & 0 \\ c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix} \tag{86}$$

be positive."

Nearly twenty years later Grommer [27] showed that if

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$$

is an entire function of null genus and we write the infinite matrix (86) which corresponds to it, then the positiveness of all of the principal minors of this function is a necessary and sufficient condition that all the roots of $f(z)$ lie in the left half-plane $\Re z < 0$. Grommer's result like Fujiwara's was in error; however, the error in his reasoning is easily removed. [1)]

We now have at our disposal all the information necessary to answer the question: *What is the largest class of entire real functions* $f(z)$ *(of any finite or infinite genus) for which there corresponds a matrix* (86) *with positive successive principal minors?*

Indeed, if we set

$$g_1(z) = c_0 + c_2 z + c_4 z^2 + \cdots$$

$$h_1(z) = c_1 + c_3 z + c_5 z^2 + \cdots$$

then according to formulas (84) the bézoutiant

$$B(g_1, h_1; \xi_0, \xi_1, \cdots) \tag{87}$$

is positive if and only if all the successive principal minors of even order of the matrix (86) are positive.

On the other hand, the bézoutiant

$$B(zh_1, g_1; \xi_0, \xi_1, \cdots) \tag{88}$$

is positive if and only if all successive principal minors of even order of the matrix

$$\begin{Vmatrix} c_0 & 0 & 0 & 0 & 0 \ldots \\ c_2 & c_1 & c_0 & 0 & 0 \ldots \\ c_4 & c_3 & c_2 & c_1 & c_0 \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

are positive. But the last mentioned minors differ from the corresponding minors of odd order one less of the matrix (86) by only the multiple $c_0 > 0$.

---

1) Grommer overlooked the fact that the functions g and h ($f = g - ih$) can have common zeros.

Therefore Hurwitz's condition (the positiveness of successive principal minors of the matrix (86)) is equivalent to the condition that both bézoutiants (87) and (88) be positive. But as we shall show directly, this is equivalent to the bézoutiant

$$B(g, h; \xi_0, \xi_1, \cdots) \tag{89}$$

of the polynomials

$$g(z) = g_1(-z^2), \quad h(z) = -zh_1(-z^2) \tag{90}$$

being positive.

In fact it follows from (90) that

$$\frac{1}{2}\frac{g(z)h(u) - g(u)h(z)}{z-u} = zu\frac{g_1(-z^2)h_1(-u^2) - g_1(-u^2)h_1(-z^2)}{u^2 - z^2} +$$

$$+ \frac{u^2 g_1(-z^2)h_1(-u^2) - z^2 g_1(-u^2)h_1(-z^2)}{u^2 - z^2}.$$

Therefore substituting in these generating functions ξ_k $(k = 0, 1, \cdots)$ for each of the powers z^k and u^k we find that

$$\frac{1}{2}B(g, h; \xi_0, \xi_1, \cdots) = B(g_1, h_1; \xi_1, -\xi_3, \xi_5, \cdots) +$$

$$+ B(zh_1, g_1; \xi_0, -\xi_2, \xi_4, \cdots).$$

Since the summands on the right-hand side contain different variables the positiveness of forms (87) and (88) implies the positiveness of form (89) and conversely.

On the other hand, it follows from (90) that

$$f(iz) = g(z) - ih(z).$$

Thus, if a real function $f(z)$ satisfies Hurwitz's conditions the function $f(iz)$ corresponds to a positive bézoutiant (89) and conversely. Therefore, we may use Theorem 11.

However, it will be more convenient for us to resort to certain considerations used in the proof of Theorem 11 rather than to use Theorem 11 immediately. In fact by virtue of these considerations, the form (89) is positive if and only if there is valid an expansion [1)]

1) We wrote e^{-2aiz} instead of $e^{-2i(z+b)}$ since $\Phi(0) = 1$ and therefore $b = 0$.

$$\Phi(z) = \frac{f(iz)}{f(-iz)} = e^{-2aiz} \prod \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\bar{z}_k}}, \tag{91}$$

where $a \geq 0$ and the roots z_k $(k = 1, 2, \cdots)$ satisfy conditions (56) and (57) of Lemma 2, and also if the points z_k are finite in number then $a > 0$.

Substituting in expansion (91) $-iz$ for z and $-i\alpha_k$ for z_k we find that

$$\frac{f(z)}{f(-z)} = e^{-2az} \prod \frac{1 - \frac{z}{\bar{\alpha}_k}}{1 + \frac{z}{\alpha_k}}.$$

Since $f(z)$ is a real function the set of points $\{\alpha_k\}$ coincides with the set of points $\{\bar{\alpha}_k\}$. Having noted this we construct an entire function $G(z)$ by the following rule

1°. If the points α_k $(k = 1, 2, \cdots)$ are of finite number m, then set

$$G(z) = e^{-az} \left[\prod_{k=1}^{m} \left(1 - \frac{z}{\alpha_k}\right) \left(1 - \frac{z}{\bar{\alpha}_k}\right) \right]^{\frac{1}{2}}. \tag{92}$$

2°. If the points α_k $(k = 1, 2, \cdots)$ are infinite in number and the sequence $\{\alpha_k\}_1^\infty$ has a finite exponent of convergence, then set

$$G(z) = e^{-az} \left[\prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) \left(1 - \frac{z}{\bar{\alpha}_k}\right) e^{2\Re\left(\frac{1}{\alpha_k^2}\right)\frac{z^2}{1} + \cdots + 2\Re\left(\frac{1}{\alpha_k^{2q}}\right)\frac{z^{2q}}{q}} \right]^{\frac{1}{2}}, \tag{93}$$

where q is the smallest number such that the series

$$\sum_{k=1}^{\infty} \frac{1}{|\alpha_k|^{2q+1}} \tag{94}$$

converges.

3°. Finally, if the points α_k $(k = 1, 2, \cdots)$ are of infinite number and the sequence $\{\alpha_k\}_1^\infty$ does not have a finite exponent of convergence, then we set

$$G(z) = e^{-az} \left[\prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) \left(1 - \frac{z}{\bar{\alpha}_k}\right) e^{2\Re\left(\frac{1}{\alpha_k^2}\right)\frac{z^2}{1} + \cdots + 2\Re\left(\frac{1}{\alpha_k^{2k}}\right)\frac{z^{2q}}{k}} \right]^{\frac{1}{2}}. \tag{95}$$

In order to convince ourselves that products (93) and (95) converge in the

appropriate cases we note that

$$\log\left[\left(1-\frac{z}{\alpha_k}\right)\left(1-\frac{z}{\bar{\alpha}_k}\right)\right]=-2\sum_{m=1}^{\infty}\frac{z^m}{m}\,\Re\left(\frac{1}{\alpha_k^m}\right),$$

and that according to Lemma 2, the series

$$C_m=\sum_{k=m}^{\infty}\Re\left(\frac{1}{\alpha_k^{2m+1}}\right)=(-1)^{m+1}\sum_{k=m}^{\infty}\Im\left(\frac{1}{z_k^{2m+1}}\right),$$

and also for any complex z the double series

$$P(z)=\sum_{m=1}^{\infty}C_m z^{2m+1}$$

are absolutely convergent.

It is easy to see that the entire function $G(z)$ which has been constructed has the property that

$$\frac{G(z)}{G(-z)}=e^{-2\varphi z}\prod\frac{1-\frac{z}{\alpha_k}}{1+\frac{z}{\alpha_k}},$$

and also the property that every zero of the function $G(z)$ is a zero of the function $f(z)$. Therefore, f/G is an entire and, moreover, even function. Therefore the equality

$$f(z)=F(z^2)\,G(z)$$

is valid, where $E(z)$ is some real entire function.

In order to formulate the result we have obtained more clearly, we introduce the concept of types $(\mathfrak{G}_p)$ $(p=0,\ 1,\ 2,\cdots,\ \infty)$.

We shall say that an entire function $G(z)$ belongs to type $(\mathfrak{G}_0)$ if it admits a representation (92) in which

$$a>0,\ \Re\alpha_k<0\quad(k=1,\ 2,\cdots,\ m);$$

to type $(\mathfrak{G}_q)$ $(q=1,\ 2,\cdots)$, if it admits a representation (93) in which

$$a\geq 0,\ \Re\alpha_k<0\ (k=1,\ 2,\cdots)\ \text{and}\ \sum_{k=1}^{\infty}\Re\left(-\frac{1}{\alpha_k}\right)<\infty,\tag{96}$$

and q is the smallest integer for which series (94) converges, and to type $(\mathfrak{G}_\infty)$

if it admits a representation (95) in which conditions (96) are satisfied but series (94) diverges for any q.

Theorem 12. *In order for an entire real function*

$$f(z) = c_0 + c_1 z + \cdots \quad (c_0 > 0)$$

to admit a representation

$$f(z) = E(z^2)\, G(z),$$

where $G(z)$ is an entire function of one of the types $(\mathfrak{G}_q)$ $(q = 0, 1, 2, \cdots)$ and $E(z)$ is an arbitrary real entire function, it is necessary and sufficient that the function $f(z)$ satisfy Hurwitz's conditions, i.e., that the consecutive principal minors of the matrix

$$\begin{Vmatrix} c_1 & c_0 & 0 & 0 & 0 & \cdots \\ c_3 & c_2 & c_1 & c_0 & 0 & \cdots \\ c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix}$$

be positive.

The theorem takes a strikingly simple form if we restrict our attention to functions of not higher than first genus. We shall formulate this as a corollary to Theorem 12.

Corollary 1. *In order that an entire real function*

$$f(z) = c_0 + c_1 z + \cdots \quad (c_0 > 0)$$

of not higher than first genus admit a representation

$$f(z) = E(z^2)\, e^{-az} \left[\Pi \left(1 - \frac{z}{\alpha_k}\right) \left(1 - \frac{z}{\bar{\alpha}_k}\right) \right]^{\frac{1}{2}},$$

where $\Re \alpha_k < 0$ $(k = 1, 2, \cdots)$, $a \geq 0$ (the equal sign holds if the α_k are finite in number), and $E(z)$ is an arbitrary real entire function (of course of not higher than first genus), it is necessary and sufficient that the function $f(z)$ satisfy Hurwitz's conditions.

Since $a > 0$, if the roots α_k are finite in number it follows that a function of null genus which satisfies Hurwitz's conditions always has an infinite set of

zeros in the left half-plane.

From Theorem 11, as we shall see, it does not follow at all that the roots of an entire real function $f(z)$ which satisfies Hurwitz's conditions lie within the left half-plane. Still, we may assert that together with every root α lying in the right half-plane $\Re z \geq 0$ the function $f(z)$ also has roots $-\alpha$, $\overline{\alpha}$ and $-\overline{\alpha}$ (since α^2 and $\overline{\alpha}^2$ are in this case roots of the function $E(z)$).

Therefore, if it is known in advance that a function $f(z)$ does not have roots which differ in sign only, then from Hurwitz's conditions it does indeed follow that all the roots of the function $f(z)$ lie in the left half-plane.

In the somewhat more special case in which among the roots of $f(z)$ there are none equal in modulus and not conjugate, interesting results to the question of determining how many of the first m in modulus of the roots of $f(z)$ lie in the left half-plane were obtained by N. G. Čebotarev in his paper cited repeatedly.

We also comment that our methods make it possible to give an exhaustive characterisation of meromorphic functions $f(z)$ which satisfy Hurwitz's conditions. However, this is hardly of interest and we shall not present it.

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